Discrete Geometric Optimal Control of Multi-body Systems

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Summary. This paper studies optimal control of multi-body systems by constructing numerical methods operating intrinsically in the state space manifold and exploiting its Lie group structure. The goal is to avoid issues with singularities related to local coordinates and to achieve efficiency through more accurate discrete trajectory representation in terms of sequences of flows along vector fields. This is accomplished by defining the multi-body dynamics as a differential equation on a Lie group and constructing geometric integrators as a basis for numerical optimal control. Standard nonlinear programming methods such as direct transcription can then be applied by performing differential operations in the Lie algebra. The resulting algorithms are generally applicable to dynamical systems on Lie groups and are illustrated with robotic vehicle examples.

Continuous Optimal Control Formulation

We consider the optimal control of articulated multi-body systems, expressed through the standard optimization problem

$$\text{minimize} \quad J(u) = \varphi(q(t_f), \dot{q}(t_f), t_f) + \int_{t_0}^{t_f} L(q, \dot{q}, u, t)dt \qquad \text{subject to} \quad M(q)\ddot{q} + b(q, \dot{q}) = B(q)u, \tag{1}$$

where q are local coordinates describing the system pose and joint angles, M is the mass matrix, b is the bias term, B is the control input transformation matrix and u are the control inputs that are typically restricted to $u(t) \in U$. The trajectory cost is defined by L and the terminal cost is ψ. The system can also be subject to final state constraints $\psi(q(t_f), \dot{q}(t_f), t_f) \leqslant 0$. Unlike standard methods which employ local coordinates, our approach is to define and solve the optimization problem intrinsically in the state space manifold of the system. In particular, the configuration space can be decomposed as $Q = G \times \mathbb{M}$, where G is a Lie group such as the Euclidean group $G = SE(3)$ and the vector space $\mathbb{M} \subset \mathbb{R}^\ell$ denotes the *joint space* assuming there are ℓ joints. Since Q is a direct product of a Lie group G and vector space $\mathbb{M}$ then Q is a Lie group itself, also referred to as a trivial fiber bundle (i.e. there is a fiber G attached at each base point $r \in \mathbb{M}$). The configuration space dimension is denoted by $m = \dim(Q)$, where for instance $m = 6 + \ell$ when $G = SE(3)$.

The Lie group structure can be exploited [7, 8] to develop coordinate-invariant algorithms with two key benefits over standard methods: a) *avoid singularities* associated with choosing rotational coordinates such as Euler angles [4, 2]; b) achieve higher accuracy in discretization through *sequencing of discrete flows along left-invariant vector fields* rather than interpolation in a Cartesian space [5, 6]. In addition, the proposed methods achieve numerical efficiency by 1) selecting *minimal discrete representation* while 2) guaranteeing *stability of the discrete integrator*.

Lie group equations of motion. The fundamental property of Lie groups is that each tangent vector on the manifold can be generated by translating a unique tangent vector at the identity using the group operation. More formally, each vector $\dot{q} \in T_qQ$ at configuration $q \in Q$ corresponds to a unique vector $\xi \in \mathbb{R}^m$ through $\dot{q} = q\widehat{\xi}$ where the "hat" operator $\widehat{\cdot} : \mathbb{R}^m \to \mathfrak{q}$ identifies ξ with a Lie algebra element matrix $\widehat{\xi} \in \mathfrak{q}$. Here $\mathfrak{q} \equiv T_eQ$ denotes the *Lie algebra* and $e \in Q$ denotes the group identity. The "vee" map $(\cdot)^\vee : T_eQ \to \mathbb{R}^m$ is defined as the inverse of $\widehat{\cdot}$, so that $(\widehat{\xi})^\vee = \xi$. Finally, *nonholonomic constraints* are encoded by replacing the body-fixed velocities ξ with *reduced* or *pseudovelocities* $v \in \mathbb{R}^{m-k}$ which satisfy $\xi = G(q)v$ where the matrix G specifies the direction of motion allowed by the constraints. The Lie group equations of motion of a nonholonomic multi-body system are then expressed as [1]

$$\text{kinematics:} \quad \dot{q} = q \cdot [G(q)v]^{\widehat{}}, \qquad \text{dynamics:} \quad M(q)\dot{v} + b(q, v) = B(q)u \tag{2}$$

With these definitions, the state space of the system is denoted by $X = Q \times \mathbb{R}^{m-k}$ and its dimension is $\dim(X) = 2m - k \equiv n$. The algorithms developed in this work perform optimization over state trajectories $x : [0, T] \to X$ (see Figure 1). This motivates us to regard X as an abstract n-dimensional Lie group to enable a straightforward extension of existing optimal control algorithms to X based on a set of general Lie group operations defined in Table 2.

Geometric Integration on Lie groups

Denoting the state by $x = (q, v) \in X$ the dynamics can be written according to

$$\dot{x} = x\,\widehat{f}(t, x, u), \tag{3}$$

which is the generalized version of the equations of motion (2). The Lie algebra element $f(t, x, u) \in \mathbb{R}^n \sim T_eX$ is interpreted as the "body-fixed" state velocity and the product $x\,\widehat{f}$ denotes the tangent group action of x on f. A time-update $x_k \to x_{x+1}$ is performed by evolving a geodesic motion on X, i.e. a curve with constant velocity $f_k \in \mathbb{R}^n$ which approximates the continuous flow along f using the exponential map $\exp : \mathbb{R}^n \to X$ according to

$$x_{k+1} = x_k \exp(f_k), \tag{4}$$

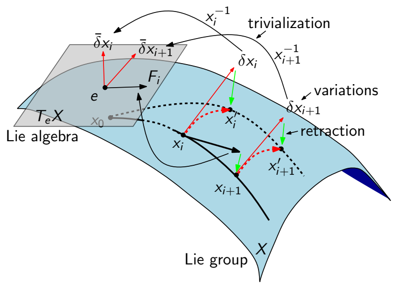


Figure 1: Trajectory variations and updates performed in the Lie algebra.

where f_k can be constructed either explicitly from x_k, u_k or could also be an implicit function of x_{k+1} as well.

[1] we purposefully swiched the role of the coordinates $q \in \mathbb{R}^m$ employed in (1) to now denote the intrinsic configuration $q \in G \times M$ used in (2). The terms M, b, B are also different are defined with respect to the new configuration q.

Operation	Vector space, $x \in \mathbb{R}^n$	Lie group, $x \in X$	Lie algebra, $\eta \in \mathbb{R}^n \sim T_e X$
Variation	$\delta x \in \mathbb{R}^n$	$\delta x \in T_x X$	$\bar{\delta} x = (x^{-1}\delta x)^\vee \in \mathbb{R}^n$
Difference	$\Delta x = x' - x \in \mathbb{R}^n$	$\Delta x = x^{-1}x' \in X$	$\bar{\Delta} x = \log(x^{-1}x') \in \mathbb{R}^n$
Increment	$x' = x + \Delta x \in \mathbb{R}^n$	$x' = x\Delta x \in X$	$x' = x \exp\left(\bar{\Delta} x\right) \in X$
Gradient	$\nabla L(x) \in \mathbb{R}^n$	$\nabla L(x) \in T_x X$	$\bar{\nabla} L(x) = \nabla_\eta L(x \exp(\eta))\vert_{\eta=0} \in \mathbb{R}^n$
ODE	$\dot{x} = f(x,u)$	$\dot{x} = x\widehat{F}(x,u)$	$\dot{\eta} = \mathrm{dexp}_{-\eta}^{-1} F(x,u),\, x = x_0 \exp(\eta)$

Figure 2: Optimal control algorithms on Lie groups can be developed analogously to standard methods in $\mathbb{R}^n$ with the help of reduced variations $\bar{\delta} x$, *left-translated* gradients $\bar{\nabla} L$, and ODEs evolving in the Lie algebra.

Discrete Optimal Control Formulation.

For numerical optimal control purposes we will employ discretized trajectories $x_{0:N} = \{x_0, ..., x_N\}$ and controls $u_{0:N-1} = \{u_0, ..., u_{N-1}\}$, where $x_k = (q_k, v_k)$. The discrete optimal control problem is formulated by minimizing

$$J(u_{0:N-1}) = L_N(x_N) + \sum_{k=0}^{N-1} L_k(x_k, u_k), \qquad \text{subject to:} \quad c_k(x_{k+1}, x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \tag{5}$$

where L_k is the total cost during time-segment $[t_k, t_{k+1}]$ for $k = 0, \dots, N-1$ and L_N is the terminal cost. The constraints c_k encode the discrete dynamics generally expressed as the implicit variable-time stepping scheme

$$c_k(x_k, x_{k+1}, u_k) = \begin{bmatrix} \log(q_k^{-1} q_{k+1}) - \frac{h_{k+1}}{2}\left[G(q_k) + G(q_{k+1})\right] v_{k+1} \\ M_{k,k+1} v_{k+1} - M_{k-1,k} v_k + \frac{1}{2}\left[h_k b(q_k, v_k) + h_{k+1} b(q_k, v_{k+1})\right] - h_{k+\frac{1}{2}} B(q_k) u_k \end{bmatrix} = 0, \tag{6}$$

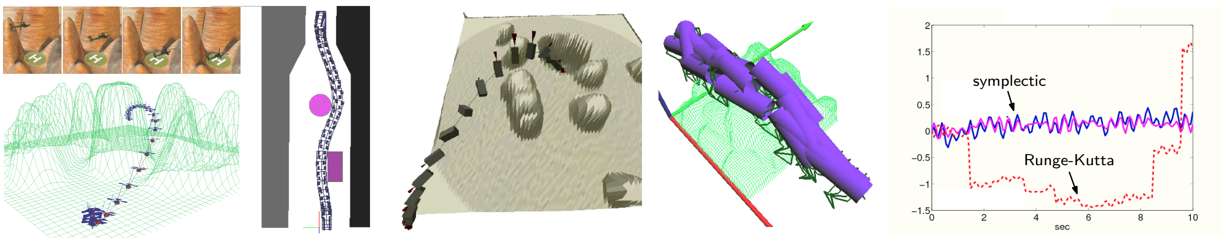
where $h_{k+1} \triangleq t_{k+1} - t_k$, $h_{k+\frac{1}{2}} \triangleq \frac{h_k + h_{k+1}}{2}$, and $M_{k,k+1} \triangleq \frac{1}{2}[M(q_k) + M(q_{k+1})]$. The relationship (6) is a particular example of the integrator (4) and is also consistent with variable time-step symplectic variational integrators [3].

Implementation: sparse nonlinear programming

Standard optimal control methods such as sequential quadratic programming (SQP) [1] are directly applicable by employing reduced/trivialized operations on Lie group elements as defined in Table 2. Denote the optimization parameter by $y = (x_{1:N}, u_{0:N-1}) \in X^N \times \mathbb{R}^{Nc}$ and its variation by $\bar{\delta} y = (\bar{\delta} x_{1:N}, \delta u_{0:N-1}) \in \mathbb{R}^{N(n+c)}$. Assume that all constraints can be encoded through the equalities $C \triangleq [c_0, c_1, \cdots, c_{N-1}] = 0$ while the boundary constraint and any other constraint using the inequalities $b(y) < 0$. Nonlinear programming problems are typically formulated by adjoining the constraint to the cost $J(y)$ using the Lagrangian $\mathcal{L}(y, \lambda) = J(y) - \lambda^T C(y) - \mu^T b(y)$ and solving the sparse QP subproblem

$$\min_{\bar{\delta} y} \bar{\nabla} \mathcal{L}(y)^T \bar{\delta} y + \frac{1}{2} \bar{\delta} y^T \bar{\nabla}_{yy} \mathcal{L}(y, \lambda) \bar{\delta} y, \qquad \text{subject to: } C(y) + \bar{\nabla} C(y)^T \bar{\delta} y = 0, \quad b(y) + \bar{\nabla} b(y)^T \bar{\delta} y \geqslant 0, \tag{7}$$

iteratively by computing the direction $\bar{\delta} y$ which is then used to update the next iterate y' either according to $x_k' = x_k' \exp(\bar{\delta} x)$ or using the updated controls $u_{0:N-1}'$ to update $x_{0:N}'$ using the nonlinear dynamics. Figure 3 shows several computed examples employing the symplectic discrete dynamics formulation defined in (6).



helicopter crossing a canyon | car and obstacles | asteroid sampling using a cubesat | legged robot on rough terrain | energy error (large time-steps)

Figure 3: Examples of optimal control of various simulated underactuated vehicles using symplectic Lie group methods. Such optimization can be performed in a few milliseconds (single-body systems) to a few seconds (18 DOF multi-body systems with simplistic contact models).

References

[1] John Betts. Survey of numerical methods for trajectory optimization. *Journal of Guidance, Control, and Dynamics*, 21(2):193–207, 1998.

[2] Elena Celledoni and Brynjulf Owren. Lie group methods for rigid body dynamics and time integration on manifolds. *Comput. meth. in Appl. Mech. and Eng.*, 19(3,4):421–438, 2003.

[3] E. Hairer, Ch. Lubich, and G. Wanner. *Geometric Numerical Integration*. Number 31 in Springer Series in Computational Mathematics. Springer-Verlag, 2006.

[4] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett, and A. Zanna. Lie group methods. *Acta Numerica*, 9:215–365, 2000.

[5] M. Kobilarov and J. Marsden. Discrete geometric optimal control on Lie groups. *IEEE Transactions on Robotics*, 27(4):641–655, 2011.

[6] M. Leok. *Foundations of computational geometric mechanics*. PhD thesis, California Institute of Technology, 2004.

[7] A. Muller and Z. Terze. Differential-geometric modelling and dynamic simulation of multibody systems. *Journal for Theory and Application in Mechanical Engineering*, 51(6), 2009.

[8] F.C. Park, J.E. Bobrow, and S.R. Ploen. A lie group formulation of robot dynamics. *The International Journal of Robotics Research*, 14(6):609–618, 1995.