Reduced Discrete Variational Multi-body Mechanics

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Abstract

This paper derives a numerical method for simulating multi-body mechanics using reduced discrete variational principles. Motivated by the favorable nature of symplectic integrators for multi-body systems [1, 2, 3, 4] we construct, for the first time, integrators defined through reduced variational principles. Then we employ the framework to compute motion controls using discrete optimal control in shape-space to satisfy boundary conditions of the full system.

Tree/Graph Structure. Consider a mechanical system consisting of interconnected rigid bodies indexed using the set of integers so that joint $\#j$ connect body $\#a_j$ to body $\#b_j$. Without loss of generality assume that body $i = 0$ is the reference body, i.e. it has no incoming joints.

Rigid Body Configuration Space. The configuration of body $\#i$ is denoted by $g_i \in SE(3)$, while its body-fixed velocity is denoted $\xi_i \in \mathfrak{se}(3)$ and defined by $\xi_i = g_i^{-1}\dot{g}_i$. In general, an element $g \in SE(3)$ and its inverse are defined using a matrix $R \in SO(3)$ and vector $x \in \mathbb{R}^3$ according to

$$g = \begin{pmatrix} R & x \\ 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} R^T & -R^T x \\ 0 & 1 \end{pmatrix}.$$

The velocities $\xi \in \mathfrak{se}(3)$ are identified with body-fixed angular and linear velocities denoted $\omega \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$, respectively, through

$$\xi = \begin{pmatrix} \widehat{\omega} & v \\ 0 & 0 \end{pmatrix}, \text{ where } \widehat{\omega} = \begin{bmatrix} 0 & -w^3 & w^3 \\ w^3 & 0 & -w^1 \\ -w^2 & w^1 & 0 \end{bmatrix}.$$

The algorithm is thus implemented in terms of vectors in $\mathbb{R}^6$ rather than matrices in $\mathfrak{se}(3)$. In the sequel we will use the following shorthand notation $g_{0:N} = \{g_0, ..., g_N\}$ to denote the configurations of all bodies of the multi-body system.

The inertia tensor of the body is denoted by $\mathbb{J}_i : \mathfrak{se}(3) \to \mathfrak{se}(3)^*$ and implemented as a 6x6 diagonal matrix. Each body is subject to potential energy, e.g. due to gravity, defined by the function $V : SE(3) \to \mathbb{R}$.

Joints. The system has n joints described by parameters $r \in M$, where $M \subset \mathbb{R}^n$ is the *shape space*. The relative transformation between the reference body configuration g and the configuration of body#i is denoted by $g_{0i} : M \to$ SE(3). We assume that all joints are controlled using control inputs denoted by $f \in \mathbb{R}^m$.

Discrete Trajectory. The trajectory of a rigid body over the time interval $[0, T]$ is represented numerically using a set of $K + 1$ equally spaced in time points denoted $g_{0:K} := \{g_0, ..., g_K\}$, where $g_k \approx g(kh)$ and $h = T/K$ denotes the time-step. The curve segment between each pair of points g_k and g_{k+1} is interpolated by a short curve that must lie on the manifold $SE(3)$. The simplest way to construct such a curve is through the map τ and a vector $\xi_k \in \mathfrak{se}(3)$ such that $\xi_k = \tau^{-1}((g_k)^{-1}g_{k+1})/h$. Such a discretization can be regarded as a discrete approximation of the continuous curve according to

$$g(t) \approx g_k \tau\left((t - hk)\xi_k\right), \qquad r(t) \approx r_k + \frac{t - hk}{h}\left(r_{k+1} - r_k\right), \text{ for } t \in [kh, (k+1)h].$$

Reduced Lagrangian. The dynamics will be derived using the reduced Lagrangian [5, 6, 4] $\ell : G \times \mathfrak{g} \times TM \to \mathbb{R}$ defined by

$$\ell(g, \Omega, r, \dot{r}) = \frac{1}{2}\begin{bmatrix} \Omega \\ \dot{r} \end{bmatrix}^T \begin{bmatrix} \mathbb{I}(r) & 0 \\ 0 & m(r) - \mathcal{A}(r)^T \mathbb{I}(r)\mathcal{A}(r) \end{bmatrix} \begin{bmatrix} \Omega \\ \dot{r} \end{bmatrix} - V(g, r) \tag{1}$$

where $\mathcal{A} : TQ \to \mathfrak{g}$ is the mechanical connection defined by $g^{-1}\dot{g} = \Omega - \mathcal{A}(r)\dot{r}$ and $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is the locked inertia tensor. Regarding them as matrices they are computed according to

$$\mathbb{I} = \mathbb{I}_0 + \sum_{i=1}^{n} A_i^T \mathbb{I}_i A_i \quad , m = \sum_{i=1}^{n} J_i^T \mathbb{I}_i J_i, \quad \mathcal{A} = \mathbb{I}^{-1} \left(\sum_{i=1}^{n} A_i^T \mathbb{I}_i J_i \right), \tag{2}$$

using the adjoint notation $A_i := \mathrm{Ad}_{g_{0i}^{-1}(r)}$, and Jacobian $J_i := g_{0i}^{-1}(r)\partial_r g_{0i}(r)$.

Discrete Variational Principle. With a discrete trajectory and a Lagrangian in place we construct a structure-preserving (i.e. group, momentum, and symplectic) integrator of the dynamics. This is accomplished by defining a discrete variational principle through a trapezoidal quadrature approximation and, using the shorthand

$$\ell_{k+\alpha} := (1-\alpha)\ell(g_k, \Omega_k, r_k, u_k) + \alpha\ell(g_{k+1}, \Omega_k, r_{k+1}, u_k), \qquad \mathcal{A}_{k+\alpha} := (1-\alpha)\mathcal{A}(r_k) + \alpha\mathcal{A}(r_{k+1}), \tag{3}$$

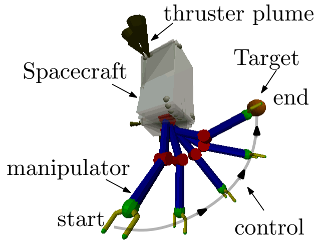
results in the discrete equations of motion (see [4])

$$(d\tau^{-1}_{\Omega_k - \mathcal{A}_{k+\alpha}u_k})^* \mu_k - (d\tau^{-1}_{-\Omega_{k-1}+\mathcal{A}_{k-1+\alpha}u_{k-1}})^* \mu_{k-1} = -h g_k^* \partial_g V(g_k, r_k), \tag{4}$$

$$(p_k - p_{k-1})/h - (1-\alpha)\langle \mu_k, \partial\mathcal{A}(r_k)(\cdot, u_k)\rangle - \alpha \langle \mu_{k-1}, \partial\mathcal{A}(r_k)(\cdot, u_{k-1})\rangle = \left((1-\alpha)\partial_r \ell_k^- + \alpha\partial_r \ell_{k-1}^+\right) + f_k, \tag{5}$$

where $\Omega_k := \mathbb{I}^{-1}\mu_k$ and $u_k := \partial_u \ell^*_{k+\alpha}(p_k - \mathcal{A}^*_{k+\alpha}\mu_k)$. The discrete Euler-Poincare equation (4) and the discrete reduced Euler-Lagrange equation (5) are implicit equations in terms of the unknown momenta μ_k and p_k. In addition, once (4)-(5) are solved the updated configurations are computed according to

$$r_{k+1} = r_k + h u_k, \qquad g_{k+1} = g_k \tau\left(h\left(\Omega_k + ((1-\alpha)\mathcal{A}(r_k) + \alpha\mathcal{A}(r_{k+1}))\, u_k\right)\right). \tag{6}$$



Computed Example . Consider a robotic satellite equipped with a 3-dof manipulator with two joints: 2-dof shoulder and 1-dof elbow. The general task is to compute a minimum control effort trajectory reaching a desired state. The algorithm is implemented and simulated as shown on left.

Motion Control. The reduced formulation enables a natural approach to motion control. Imagine that the free-floating manipulator must grasp a particular target in task-space, or equivalently to move between two given zero-velocity states $(g_0, 0, r_0, 0)$ and $(g_N, 0, r_N, 0)$. The approximate solution shown above was computed by performing nonlinear optimization of shape trajectories $r_{0:K}$ and by reconstructing the full state $(g_{k+1}, \xi_k, r_{k+1}, u_k)$ for each k using (6). The objective function includes the control effort as well as a final cost pelanizing deviation from the given boundary conditions.

Motion optimization based on the mechanical connection is efficiently implemented using a stage-wise Newton method [7] which has complexity linear in the number of discrete segments K. The algorithm requires first-order derivatives of the equations of motion (4)-(6) and first and second-order derivatives of the given cost function.

References

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