Trajectory tracking of a class of underactuated systems with external disturbances

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***Abstract*— This paper considers trajectory tracking of vehicles modeled as underactuated rigid bodies. The goal is to obtain a simple and intuitive convergent controller that avoids issues with tuning gains or using coordinates associated with singularities. The main novelty lies in the derivation of a generally applicable control law that handles external disturbances.**

I. Introduction

This paper studies trajectory tracking for mechanical systems modeled as rigid bodies with full attitude control and one translational force input. Typical applications include control of underactuated aerial or underwater vehicles and nanosatellites. This is a classical problem in underactuated mechanics and has received much attention, especially recently with the popularization of aerial vehicles such as quadcopters. Despite being a very specific problem, a plethora of different control methods have been proposed and there does not seem to be a consensus about the most appropriate and generally applicable approach. The purpose of this work is to propose a simple yet general method equipped with stability guarantees and try to clarify its advantages and limitations in context of existing techniques.

Tracking for an underactuated rigid body is a nontrivial problem. First, due to underactuation only a subset of the degrees of freedom can be freely specified. It has been shown that controlling the position and the angle around the translation force input axis of the vehicle is a choice that does not result in unstable zero dynamics [1]. Choosing these coordinates as outputs then renders the system feedback linearizable and appropriate virtual controls are found using dynamic decoupling [2] (or equivalently known as dynamic extension [3]). For the system considered in this paper this procedure entails differentiating the equations of motion twice and inserting integrators in front of the thrust input. Such a feedback linearization approach has also more recently been applied to quadrotor vehicles [4], [5], [6]. Special structure in the system can be exploited for better efficiency (e.g. lower torque and thrust) by methods other than feedback linearization alone. In particular, backstepping is used to successively stabilize parts of the system enabling more efficient feedback loop closing in stages. A number of methods have been recently proposed for controlling aerial vehicles using backstepping. The majority of these works are based on local coordinate transformations and successive coordinate-wise splitting and backstepping procedures [7], [4], [8], [9], [10], [11], [12], [13]. A subset of these developments also note that backstepping results in improved disturbance rejection in the presence of noise [14], [10], [6], [12]. In addition, the effects of control bounds have also been formally considered [15]. Backstepping method for tracking on manifolds [16] have been proposed to deal with coordinate singularities and achieve almost globally stable behavior. Other related methods for tracking on manifolds have also been developed [17], [18], [19] that result in simpler control laws but rely on stronger assumptions. Many of these methods have also been implemented successfully on a number of real vehicles.

Finally, we note that the area of unmanned vehicles is rapidly growing and many new improvements in both hardware and control are possible. Most recent examples, among many, include achieving precise control through iterative refinement based on experiments [20] or employing variable-pitch rotors for greater stability and control responsiveness [21].

The paper is organized as follows. The problem is formulated in §II and the tracking control along with stability guarantees for the nominal disturbance-free system are given in §III and for the full system in §IV. Applications to a simulated aerial vehicle and a simulated nanosatellite are presented in §V.

II. Problem Formulation

The vehicle is modeled as a single underactuated rigid body with position $p \in \mathbb{R}^3$ and orientation matrix $R \in \mathrm{SO}(3)$. The *body-fixed* angular velocity is denoted by $\omega \in \mathbb{R}^3$. The vehicle has mass m and rotational inertia tensor $\mathbb{J}$. The state space of the vehicle is $S = SE(3) \times \mathbb{R}^6$ with $s = ((R, p), (\omega, \dot{p})) \in S$ denoting the whole state of the system.

The vehicle is actuated with control torques $\tau \in \mathbb{R}^3$ and a control force $u > 0$ applied in a body-fixed direction defined by the unit vector $e \in \mathbb{R}^3$. The vehicle is subject to known external forces and torques denoted by the functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and $f_\omega : S \to \mathbb{R}^3$, respectively. In addition, it is subject to disturbances $\delta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\delta_\omega : S \to \mathbb{R}^3$ which are unknown but bounded according to

$$\|\delta(p, \dot{p})\| \leqslant \rho(p, \dot{p}), \qquad \|\delta_\omega(s)\| \leqslant \rho_\omega(s), \tag{1}$$

for some known positive functions ρ and ρ_ω. The equations of motion are

$$m\ddot{p} = Reu + f(p, \dot{p}) + \delta(p, \dot{p}), \tag{2}$$

$$\dot{R} = R\hat{\omega}, \tag{3}$$

$$\mathbb{J}\dot{\omega} = \mathbb{J}\omega \times \omega + \tau + f_\omega(s) + \delta_\omega(s, \tau), \tag{4}$$

where the map $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{5}$$

Tracking for the nominal disturbance-free system is studied first, after which the full model is considered.

The Tracking Problem: The goal is to track a trajectory specified by the functions $(R_d, p_d) : [0, T] \to SE(3)$. Motivated by considerations noted in earlier work employing feedback linearization our proposed approach is concerned with controlling only the position p and one degree of freedom of the orientation R, i.e. rotations around the axis e. For clarity, our development will focus on position tracking only. Controlling the additional rotational degree of freedom can be accomplished in the same framework by augmenting the Lyapunov-based design with terms encoding either desired Euler angles [7] or rotation matrices directly [16].

III. Tracking for the nominal system

We first consider the simpler case without state-dependent external forces and without disturbances, i.e. when

$$f(p, \dot{p}) \equiv f = a_g m, \;\; \delta(p, \dot{p}) = 0, \;\; f_\omega(s) = 0, \;\; \delta_\omega(p, \dot{p}) = 0,$$

where $a_g \in \mathbb{R}^3$ is the gravity acceleration vector. In order to simplify the control law design, the nominal dynamics are expressed according to

$$\dot{x} = Ax + B[f + g(R, u)], \tag{6}$$

$$\dot{R} = R\hat{\omega}, \tag{7}$$

$$\dot{\omega} = \mathbb{J}^{-1}\left[\mathbb{J}\omega \times \omega + \tau\right], \tag{8}$$

where $x \in \mathbb{R}^6$ denotes the state $x = (p, \dot{p})$, the control vector $g : SO(3) \times \mathbb{R} \to \mathbb{R}^3$ is defined by $g(R, u) = Reu$, and the matrices A and B are given by

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} I \end{bmatrix}. \tag{9}$$

The term $g(R, u)$ is regarded as a virtual control input for the subsystem (6) with respect to the error

$$z_0(t) = x(t) - x_d(t).$$

The first step is to define the desired force g_d by

$$g_d(t, x) = m\ddot{p}_d(t) - Kz_0(t) - f, \tag{10}$$

where the gain matrix K satisfies the standard Lyapunov condition

$$P(A - BK) + (A - BK)^T P = -Q,$$

for some positive definite symmetric matrices P and Q. The associated storage function

$$V_0(t, x) = \frac{1}{2} z_0^T P z_0 \geqslant 0,$$

then evolves according to

$$\dot{V}_0 = -\frac{1}{2} z_0^T Q z_0 + (B^T P z_0)^T (g - g_d). \tag{11}$$

The next step is to define the storage function

$$V_1(t, x, R, u) = V_0 + \frac{1}{2}\|z_1\|^2 \geqslant 0,$$

where the error z_1 is defined by

$$z_1(t, x, R, u) = g(R, u) - g_d(t, x).$$

The evolution of V_1 is computed according to

$$\dot{V}_1 = \dot{V}_0 + z_1^T \left(\dot{g} - m p_d^{(3)} + K\dot{z}_0\right) \tag{12}$$

Next, the desired value of $\dot{g}$ is defined according to

$$a_d(t, x, R, u) = m p_d^{(3)} - K\dot{z}_0 - B^T P z_0 - k_1 z_1,$$

for some $k_1 > 1$. After substituting a_d in (12) we obtain

$$\dot{V}_1 \leqslant -\frac{1}{2} z_0^T Q z_0 - \frac{k_1}{2} z_1^T z_1 + z_1^T (\dot{g} - a_d).$$

Next, define the storage function

$$V_2(t, x, R, u, \omega, \dot{u}) = V_1 + \frac{1}{2}\|z_2\|^2 \geqslant 0, \tag{13}$$

where the error z_2 is defined by

$$z_2 = \dot{g} - a_d \tag{14}$$

$$= R(\hat{w}eu + e\dot{u}) - a_d. \tag{15}$$

Taking its derivative we obtain

$$\dot{V}_2 = \dot{V}_1 + z_2^T (\ddot{g} - \dot{a}_d) \tag{16}$$

The desired value of $\ddot{g}$ is defined by the vector

$$\begin{aligned} b_d &= \dot{a}_d - z_1 - k_2 z_2 \\ &= m p_d^{(4)} - K(A^2 x + ABg + B\dot{g} - \ddot{x}_d) \\ &\quad - B^T P \dot{z}_0 - k_1 \dot{z}_1 - z_1 - k_2 z_2, \end{aligned} \tag{17}$$

for some $k_2 > 0$. After substituting (17) in (16) we obtain

$$\dot{V}_2 \leqslant -\frac{1}{2} z_0^T Q z_0 - \frac{k_1}{2} z_1^T z_1 - \frac{k_2}{2} z_2^T z_2 + z_2^T (\ddot{g} - b_d) \tag{18}$$

The relationship $\ddot{g} = b_d$ can be satisfied directly by noting that

$$\ddot{g}(R, u) = R\left[\hat{\omega}^2 eu + 2\hat{\omega}\dot{u} + \dot{\omega} \times eu + e\ddot{u}\right] \tag{19}$$

and will hold true if one could choose $\dot{\omega}$ and $\ddot{u}$ to satisfy

$$\dot{\omega} \times eu + e\ddot{u} = R^T b_d - \hat{\omega}^2 eu - 2\hat{\omega}\dot{u}. \tag{20}$$

Therefore, in view of the dynamics (4), the control law

$$\begin{aligned} \tau &= \mathbb{J}[e \times (R^T b_d - \hat{\omega}^2 eu - 2\hat{\omega}\dot{u})/u] - \mathbb{J}\omega \times \omega - f_\omega(s), \\ \ddot{u} &= e^T (R^T b_d - \omega^2 eu - 2\hat{\omega}\dot{u}), \end{aligned} \tag{21}$$

results in $\ddot{g} = b_d$ and hence (18) reduces to $\dot{V}_2 \leqslant 0$ which proves that the nominal system is asymptotically stable.

To summarize, the control law (21) was designed by regarding the force vector $g(R, u)$ as a virtual input to the position dynamics and performing backstepping until the control variables $\ddot{u}$ and τ appeared explicitly and set to satisfy the asymptotic stability requirements of the full system expressed by the Lyapunov function (13) and its negative derivative (18). The actual input u will in practice be computed by integrating $\ddot{u}$.

Setting the controls prematurely.: Derivatives of the control input u appear for a reason. It is tempting to use static choice such as

$$u = (Re)^T g_d, \tag{22}$$

since it corresponds to projecting the available control vector onto the desired control vector. In general, such a choice is only possible under restrictive assumptions on the boundedness of the uncontrollable terms. Instead, such complications can be avoided by exploiting the structure of the dynamics by regarding the terms $u, \dot{u}$ as additional dynamic compensation state variables while $\ddot{u}$ becomes an input to the system. Thus, our approach builds upon works such as [1], [7], [16] to extend the control design in the presence of external disturbances.

IV. Disturbance Attenuation Control

We next consider the robust control of the full system subject to unknown disturbance forces. The system dynamics (2)-(4) are equivalently expressed according to

$$\dot{x} = Ax + B[g(R, u) + f(x) + \delta(x)] \tag{23}$$

$$\dot{R} = R\hat{\omega} \tag{24}$$

$$\dot{\boldsymbol{\omega}} = \mathbb{J}^{-1}\left[\mathbb{J}\omega \times \omega + \tau + f_\omega(s) + \delta_\omega(s)\right], \tag{25}$$

employing the definitions (9). The term $g(R, u)$ is again regarded as a virtual control input for the subsystem (23). Its desired value g_d is chosen to minimize the error

$$z_0(t) = x(t) - x_d(t).$$

and to suppress the disturbance $\delta(x)$, and is defined by

$$g_d(t, x) = \ddot{p}_d - Kz_0 - f(x) - \eta_0(x)\frac{B^T P z_0}{\|B^T P z_0\|}, \tag{26}$$

where $\eta_0(x) \geqslant \rho(x)$. Similarly to §III the gain matrix K and symmetric positive definite matrix P are chosen to satisfy the standard Lyapunov condition

$$P(A - BK) + (A - BK)^T P = -Q,$$

for some positive definite symmetric matrix Q. The storage function

$$V_0(t, x) = \frac{1}{2} z_0^T P z_0,$$

then evolves according to

$$\begin{aligned}\dot{V}_0 = &-\frac{1}{2} z_0^T Q z_0 + (B^T P z_0)^T \delta - \eta_0 \|B^T P z_0\| \\ &+ (B^T P z_0)^T (g - g_d) \\ \leqslant\ & (B^T P z_0)^T (g - g_d),\end{aligned} \tag{27}$$

where the inequality follows from the relationship

$$\|\delta\| \leqslant \rho \leqslant \eta_0.$$

The following definition will enable a compact derivation:

Definition 4.1: Let $\psi(x, \cdot)$ be a function of $x(t)$ and other time-dependent variables. The function $\bar{\psi}(x, \cdot)$ then denotes the total time-derivative of ψ with respect to all arguments but x, i.e.

$$\bar{\psi} = \dot{\psi} - \partial_x \psi \cdot \dot{x}.$$

Next, define the error

$$z_1(t, x, R, u) = g(R, u) - g_d(t, x),$$

and the storage function

$$V_1(t, x, R, u) = V_0 + \frac{1}{2}\|z_1\|^2 \geqslant 0.$$

Taking its derivative we obtain

$$\dot{V}_1 = \dot{V}_0 + z_1^T \left\{\dot{g} - \bar{g}_d - \partial_x g_d \cdot [Ax + B(f + g + \delta)]\right\} \tag{28}$$

Next, a desired value of $\dot{g}$ is defined by the vector

$$\begin{aligned}a_d(t, x, R, u) =& \bar{g}_d - B^T P z_0 - k_1 z_1 + \partial_x g_d \cdot [Ax + B(f + g)] \\ &- \eta_1(x)\frac{B^T \partial_x g_d^T z_1}{\|B^T \partial_x g_d^T z_1\|},\end{aligned}$$

for some $k_1 > 0$, $\eta_1(x) \geqslant \rho(x)$, which is substituted in (28) to obtain

$$\dot{V}_1 \leqslant -\frac{1}{2} z_0^T Q z_0 - \frac{k_1}{2} z^T z_1 + z_1^T(\dot{g} - a_d).$$

Next, define the error

$$z_2 = \dot{g} - a_d,$$

and the storage function

$$V_2(t, x, R, u, \omega, \dot{u}) = V_1 + \frac{1}{2}\|z_2\|^2 \geqslant 0.$$

Taking its derivative we obtain

$$\dot{V}_2 = \dot{V}_1 + z_2^T \left\{\ddot{g} - \bar{a}_d - \partial_x a_d \cdot [Ax + B(f + g + \delta)]\right\} \tag{29}$$

The desired value of $\ddot{g}$ is defined according to

$$\begin{aligned}b_d =& \bar{a}_d - z_1 - k_2 z_2 + \partial_x a_d \cdot [Ax + B(f + g)] \\ &- \eta_2(x)\frac{B^T \partial_x a_d^T z_2}{\|B^T \partial_x a_d^T z_2\|}\end{aligned}$$

for some $k_2 > 0$, $\eta_2(x) > \rho(x)$, which is substituted in (29) to obtain

$$\dot{V}_2 \leqslant -\frac{1}{2} z_0^T Q z_0 - \frac{k_1}{2} z_1^T z_1 - \frac{k_2}{2} z_2^T z_2 + z_2^T(\ddot{g} - b_d) \tag{30}$$

The last term on the right-hand side in (30) can be rendered negative by directly setting the control variables τ and $\ddot{u}$. Employing the relationships (19)-(20) this is accomplished by the control law

$$\begin{aligned}\tau &= \mathbb{J}[e \times (R^T b_d - \hat{\omega}^2 eu - 2\hat{\omega}\dot{u})/u] - \mathbb{J}\omega \times \omega - f_\omega(s) + \nu, \\ \ddot{u} &= e^T(R^T b_d - \omega^2 eu - 2\hat{\omega}\dot{u}),\end{aligned} \tag{31}$$

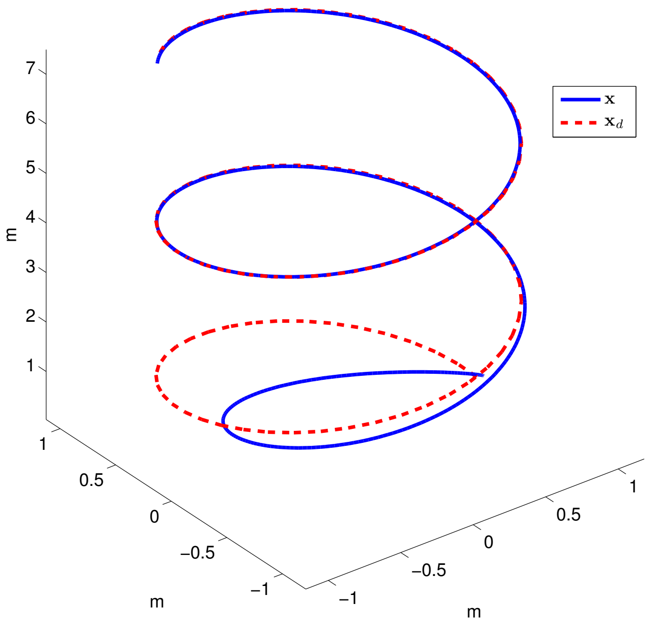

Fig. 1. An example of quadrotor path tracking.

where ν is the disturbance rejection term defined by

$$\nu = \eta_3(x)\frac{\mathbb{J}^{-1}\hat{e}R^T z_2}{\|\mathbb{J}^{-1}\hat{e}R^T z_2\|},$$

for some $\eta_3 \geqslant \rho_\omega$. The controls (31) result in $\dot{V}_2 \leqslant 0$ which proves that the full system is asymptotically stable.

Singularities.: Note that the control law is undefined when the disturbance rejection denominator terms evaluate to zero, i.e. when $\|B^T P z_0\| = 0$, $\|B^T \partial_x g_d^T z_1\| = 0$, $\|B^T \partial_x a_d^T z_2\| = 0$, or $\|\mathbb{J}^{-1}\hat{e}R^T z_2\| = 0$. In such cases, the whole fraction is set to zero and the control law is applied as usual. It is well known [22] that in practice this leads to chattering, a behavior that can be alleviated by a slight modification of the control law to remove singularities. While we do not provide the details, such an extension is readily applicable in our proposed setting.

V. Applications

A. Quadrotor

A standard quadrotor model is employed with weight $m = 0.5$kg and moments of inertia $\mathbb{J} = \mathrm{diag}(.003, .003, .005)$. The control axis is $e = (0, 0, 1)$ and the only external force is gravity, i.e. $f \approx (0, 0, -9.81m)$. Figure 1 shows an example reference path and the resulting stabilizing simulated vehicle trajectory. Figure 2 provides more details about the example. The tracking algorithm exhibits good performance and is able to handle a variety of initial conditions.

B. Nanosatellite

Consider a nanosatellite modeled as a rigid body with an attitude control system (ACS) and a single thruster aligned with the body-fixed x-axis, i.e. $e = (1, 0, 0)$. The moments

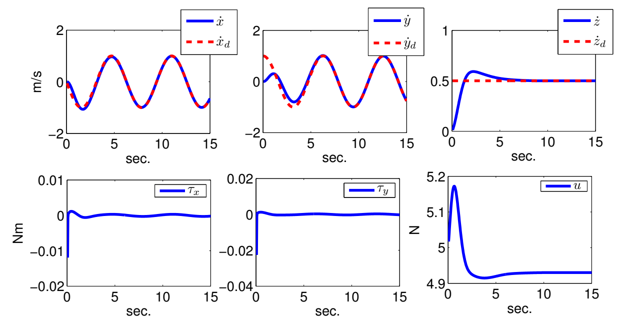

Fig. 2. History of the velocities, inputs, and Lyapunov function V of the example in Figure 1.

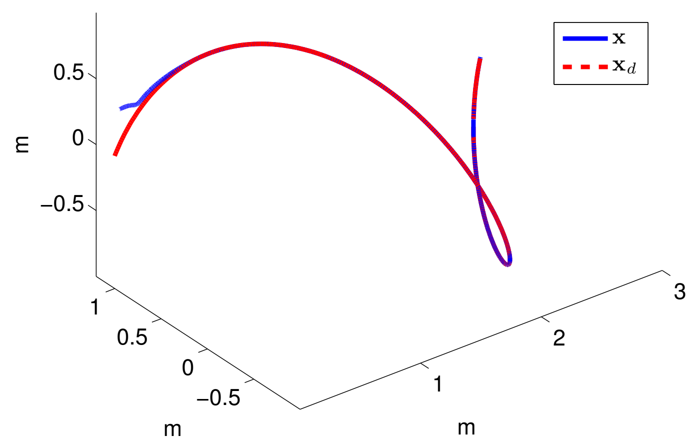

Fig. 3. An example of a nanosatellite path tracking.

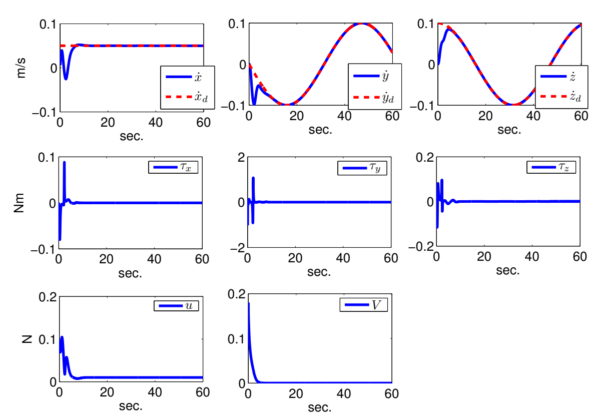

Fig. 4. History of the velocities, inputs, and Lyapunov function V of the example in Figure 3.

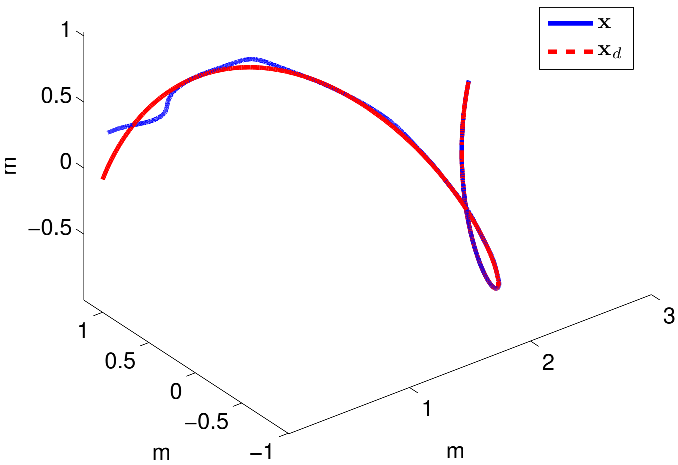

Fig. 5. An example of a nanosatellite path tracking with bounded inputs.

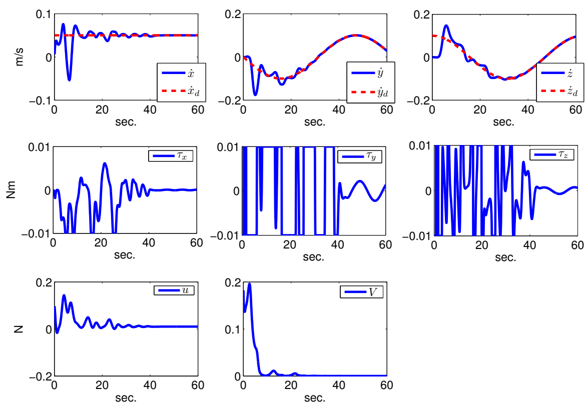

Fig. 6. History of the velocities, inputs, and Lyapunov function V of the example in Figure 5.

of inertia of the satellite are $\mathbb{J} = \mathrm{diag}(.03, .03, .015)$. No external forces are included in these simulations. Figure 3 shows an example reference path and the resulting stabilizing simulated vehicle trajectory.

While the controller exhibits good performance (Figure 4) it is immediately obvious that the resulting control inputs u and $\boldsymbol{\tau}$ are unrealistic for a small satellite such as a cubesat. In practice, the available torque from reaction wheels and magnetorquers is very limited, typically on the order of 10mN-meter. Thus, a more realistic application of the the exact same controller but with bounded inputs is simulated and shown on Figures 5 and 6. Under such tight bounds, stability in its strict sense is no longer guaranteed which is also evident from the bumps in the Lyapunov function curve. Yet, the controller achieves stabilization and exhibits a convergent overall trend.

VI. Conclusion

Trajectory tracking for underactuated systems has been previously addressed through a variety of methods. The proposed approach employs backstepping and Lyapunov redesign to handle disturbances in a coordinate-free setting. Empirical evidence suggests that the approach can be used as a basis for developing methods to handle tight actuator bounds. Future work will consider stability guarantees in such cases.

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