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Discrete mechanics optimal control (DMOC) and model predictive control (MPC) synthesis for reaction-diffusion process system with moving actuator

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Abstract— **This paper is concerned with the computation of optimal motion control as well as the optimal input injection policy of an actuator arm regulating the temperature in a reaction-diffusion system. The system has two dynamical components consisting of the arm mechanics with inertial, elastic and damping properties, which is driven by bounded mechanical actuation controls and an underlying reaction-diffusion system described by the parabolic PDE. The state of the actuator arm parametrizes the input injection operator of the parabolic PDE systems model and causes coupling between the two dynamical systems generally operating at different time scales. The method proposed in this paper is aimed at solving this coupled problem. The actuator mechanics and its control are achieved in the discrete mechanics and optimal control (DMOC) framework, while the input injection for the reaction diffusion system is calculated by the modal model predictive control (MMPC) algorithm suitable for the dissipative systems. The actuation arm policy and input to the parabolic PDE system include in its realization low-order discrete representation of the parabolic PDE evolution and incorporate optimality with respect to both the state of the PDE and the actuator displacement cost from current to some more optimal control position as well as naturally present input and PDE state constraints. The proposed actuation arm policy and optimal stabilization of the unstable reaction-diffusion system in the presence of constraints in the full state-feedback controller realization have been evaluated through simulations.**

Key words: Distributed-Parameter Systems, Moving Actuator, Discrete Mechanics Optimal Control (DMOC), Model Predictive Control (MPC), Input/State Constraints

I. INTRODUCTION

This work considers the control design of a temperature regulating actuator mounted over a catalytic bar in a multiscale mechanical and reaction-diffusion system. The controller synthesis is based on two different types of dynamics present – the reaction-diffusion system dynamics describing the catalytic rod temperature in terms of a parabolic PDE, and the rigid body dynamics of the actuator arm. This coupled dynamics system configuration is common in the process industry, in particular in material processing plants in which the welding arm is crossing over the welded material, or in the wood processing plants in which drying of the wood boards takes place by moving the actuator arm that provides dry air steam.

While the problem of temperature control in reaction-diffusion systems described by the parabolic PDEs is well studied [1], [2], [3], the issue of the moving actuator incorporated in the temperature and/or concentration regulator synthesis has not been explored due to the complexity of integrating mechanical system control realization with the dynamics emerging from the transport-reaction models. In other words, the complexity arising from the coupling of the finite dimensional system representation due to the mechanical system dynamics with the infinite-dimensional representation of dissipative reaction-diffusion type system's description has not been successfully resolved from the point of unifying controller synthesis. In addition to this challenging task, the natural presence of constraints which arise either from limited actuator ability (not only related to the actuation associated with the reaction-diffusion subsystem but also the one present in the mechanical subsystem) and/or stringent process specification requirements, contribute to the complexity of controller synthesis. Moreover, the requirements for optimality and natural presence of constraints further complicate the problem which at present cannot be handled systematically by any available methods. Therefore, in order to address all the aforementioned issues, we propose a novel controller synthesis approach that combines recent advances in *discrete mechanics* [4] for computational modeling of mechanical systems, and in the synthesis of model predictive control realizations suitable for *parabolic systems* [5].

We are interested in the following optimal control problem: "*Compute the actuator force $f(t)$ over a given finite time interval which brings the system to a desired state while minimizing a user-specified cost function combining the energy required for heating/cooling injected by the actuator $u(t)$, as well as the actuator control effort, subject to the dynamics of both reaction-diffusion system and mechanical actuator, and subject to the reaction-diffusion systems' state and/or input injection constraints, and actuator constraints.*" More formally,

$$\min_{u(t)\ f(t)} \int_0^T (\|y(t)\|_Q^2 + \|u(t)\|_R^2 + \|f(t)\|^2)dt \qquad (1)$$

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with constraints applied on both the mechanical system and parabolic PDE state $y(t)$ dynamics and the inputs $(u(t)\ f(t))$.

The subsystem reaction-diffusion dynamics describing the evolution of the PDE state, e.g. the temperature in the catalytic rod, is treated by the model reduction through Galerkin method and subsequent model modal predictive control (MMPC) formulation that includes constraints on the available input injection and on allowable temperature profile. The actuator arm dynamics is represented numerically through a discrete Lagrange-d'Alembert variational principle suitable for robust numerical integration and optimization purposes. The resulting discrete mechanical optimal control (DMOC) problem is combined with the constrained optimization structure emerging from the MPC realization. While there are standard methods to represent the actuator mechanics for numerical optimization purposes, we choose DMOC methodology because since it has proven effective for systems with holonomic and nonholonomic constraints [4], [6] which are present in realistic actuator arm models.

This work reports initial results towards incorporating actuator dynamics and control effort optimality required for actuator arm transfer policy subject to reaction-diffusion system dynamics constraints. In this way, the multiscale feature of the mechanical systems dynamics and temperature evolution of the conventional process systems are incorporated in the realizable controller synthesis.

MOVING ACTUATOR

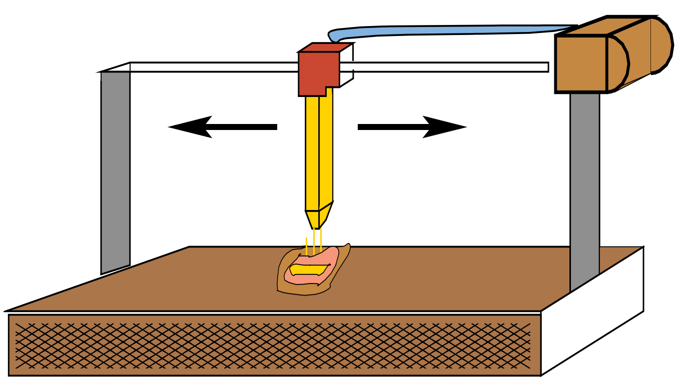

Fig. 1. Moving actuator scheme.

Dynamical model of the system is given as a multiscale system of moving actuator arm which provides a foundation for the mounted injection cooling/heating module that supplies an input to the underlying reaction-diffusion system governed by the parabolic PDE (see Fig. 1).

A simple linear actuator arm with configuration $q \in \mathbb{R}$ is modeled as,

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \tag{2}$$

where M is the arm inertia, C is damping, K is elastic constant and f is the input, e.g. from a motor, which moves the arm across the domain. This arm provides a foundation for the input which can be viewed as a heat source/sink to the reaction-diffusion system described by parabolic reaction-diffusion equation, which takes the following form,

$$\frac{\partial x}{\partial t} = \kappa \frac{\partial^2 x}{\partial \zeta^2} + \eta x + b_{(q(t))} u(t) \qquad 0 \le \zeta \le l \tag{3}$$

where x is a state variable, κ and η are constants, and $b_{(q(t))}$ provides the position of the actuator guided by the actuator sliding across the domain $[0\ l]$. Also, there are constraints present in the reaction-diffusion system

$$u_{min} \le \qquad u(t) \qquad \le u_{max} \tag{4}$$

$$x_{min} \le \quad \int_0^l \delta(\zeta_c - \zeta) x(\zeta\ t) d\zeta \quad \le x_{max} \tag{5}$$

which limit the allowable control action applied to the reaction-diffusion system and place bounds on the state applied at point ζ_c within domain. One can notice that the coupling between the two systems is through the position of the arm $q(t)$ determining the input function $b_{(q(t))} = \delta(q(t) - \zeta)$.

The problem considered is common in the process industry. A movable actuator scans the physical space domain constrained by the arm dynamics and its bounded velocity and attempts to control optimally in finite time the dynamics of the diffusion reaction system. The problem is, therefore, modeled in two layers–the mechanical system with the mass-spring-dashpod model and reaction-diffusion system modeled by the parabolic PDE. Equivalently, it is formulated as

$$\arg \min_{u(t)\ f(t)} \int_0^T (\|y(t)\|_Q + \|u(t)\|_R + \|f(t)\|)dt \tag{6a}$$

$$M\ddot{q}(t) + C\dot{q}(t) + kq(t) = f(t) \tag{6b}$$

$$\frac{\partial x}{\partial t} = \kappa \frac{\partial^2 x}{\partial \zeta^2} + \eta x + b_{(q(t))} u(t) \tag{6c}$$

$$y(t) = \mathcal{C}x(t) \tag{6d}$$

$$u^{min} \le u(t) \le u^{max} \tag{6e}$$

$$x_w^{min} \le \mathcal{S}x(t) \le x_w^{max} \tag{6f}$$

The straightforward solution by the time discretization of the mechanical system dynamics and reaction-diffusion system dynamics with the inclusion of constraints leads to complex optimization programs. However, one can explore the discrete mechanics optimal control and model predictive control methodologies in order to obtain a realizable actuation arm policy. Namely, model predictive control (MPC) [7], [8] is one of the successful engineering originated control synthesis naturally incorporating constraints that occur in practice. Such a discrete system based synthesis solves the finite horizon optimal control problem through optimization (e.g., LP or QP) achieving stabilization and constraints satisfaction over the given finite horizon. A specific feature of the model predictive control algorithm, i.e. the state constraints relaxation method [9], has already been utilized in the distributed and boundary model predictive control of parabolic systems [10], [11]. Recently, there were works that considered a number of actuators distributed along the domain and an optimal switching scheme which should be applied to spatially explore the system by switching the controller realizations [12]. However, the important notion of optimality within the moving actuator control of the parabolic PDEs setting has not been explored, nor has the issue of the point state constraints been addressed in the related work. Since the control and state/output constraints are naturally present in the engineering practice, their proper inclusion in the infinite dimensional setting of the distributed parameter systems representation is of paramount interest.

In this paper, the parabolic PDE control framework [11] is augmented with optimal control of the moving actuator arm in the DMOC setting. In particular, the evolution of the optimal control problem for the arm dynamics is solved by incorporating exact dynamics of the mechanical system, while the underlying parabolic PDE is treated within the modal model predictive control setting for the dissipative systems. The model modal predictive controller (MMPC) benefits from the dominant diagonal feature of the unbounded system's operator and from the separation among the finite number of unstable modes and the infinite dimensional stable modal complement. The proposed control problem formulation has been evaluated through simulations in the case of a full-state feedback control synthesis.

II. Preliminaries

A. Discrete Mechanics and Optimal Control

One of the primary goals of this paper is to consider the reaction-diffusion system driven by the actuator with *dynamics* which must be explicitly accounted for to guarantee feasible and optimal control design. We choose to represent computationally the actuator dynamics using the methodology originated from *discrete mechanics* and *variational integrators* [4], which has provable preservation properties leading to accurate and stable numerical schemes. The main advantage of discrete mechanics with the optimal control (DMOC) lies in its robust numerical representation of dynamics that is suitable for complex systems with constraints, such as an actuator with multi-rigid body dynamics and/or actuator with end-effector bound to a given surface in the workspace. In addition, an interesting link between asynchronous variational integrators [13] and the multi-scale nature of the system dynamics in this work will be of interest in our future work.

Assume that we are given a mechanical system on a manifold Q with Lagrangian $L : TQ \rightarrow \mathbb{R}$ and control forces $f \in T^*Q$ (here T^*Q denotes the space dual to the tangent bundle TQ consisting of the system velocity vectors). Our goal is to find a trajectory $q : [0\ \ T] \rightarrow Q$ between two given states $q(0)\ \ \dot{q}(0)$ and $q(T)\ \ \dot{q}(T)$ which minimizes a given cost function,

$$J(q\ \ f) = \int_0^T \Theta(q(t)\ \ f(t))dt$$

while satisfying the system dynamics. The cost $\Theta : T^*Q \rightarrow \mathbb{R}$ can, for instance, represent the control effort $\Theta = \frac{1}{2}\|f\|^2$ or time $\Theta = 1$. In the absence of constraints, the dynamics is derived through the *Lagrange-d'Alembert variational principle* which requires that the trajectory $q : [0\ \ T] \rightarrow Q$ satisfies

$$\delta \int_0^T L(q\ \ \dot{q})dt + \int_0^t f \cdot \delta q = 0 \tag{7}$$

This leads to the *Euler-Lagrange* equations of motion

$$\partial_t \partial_{\dot{q}} L - \partial_q L = f$$

which can be discretized using finite differences and used as constraints in nonlinear optimization of $J(q\ \ f)$.

Alternatively, the dynamics can be derived numerically by discretizing the variational principle directly. This is achieved by first discretizing the trajectory in time using a finite set of N_d points $q_0 \quad q_{N_d}$ with fixed time-step $h = T\ \ N_d$ and applying a *discrete Lagrange-d'Alembert principle*:

$$\delta \sum_{k=0}^{N_d-1} L(q_{k+\alpha}\ \ \frac{q_{k+1}-q_k}{h}) + \sum_{k=0}^{N_d-1} f_{k+\alpha} \cdot \delta q_{k+\alpha} = 0 \tag{8}$$

by varying each one of these points separately. Here, the notation

$$x_{k+\alpha} = (1-\alpha)x_k + \alpha x_{k+1} \qquad \alpha \in [0\ \ 1]$$

was used to denote an interpolated point along the line segment between x_k and x_{k+1}. The parameter α determines the quadrature point used to approximate the integrals in (7). Typical values for α are 0, 1 2, and 1. Applying the variations in (8) results in the following discrete equations of motion:

$$\begin{aligned} &\frac{1}{h}(\partial_v L_k - \partial_v L_{k-1}) - (1-\alpha)\partial_q L_k - \alpha \partial_q L_{k-1} \\ &= (1-\alpha) f_{k+\alpha} + \alpha f_{k-1+\alpha} \end{aligned} \tag{9}$$

using the shorthand notation $L_k := L(q_{k+\alpha}\ \ \frac{q_{k+1}-q_k}{h})$. The discrete equations of motion (9) can now be used to formulate a constrained optimization problem as follows:

Discrete Mechanics and Optimal Control (DMOC):

$$\min_{q_{0:N_d}\ f_{0:N_d}} \sum_{k=0}^{N_d-1} \Theta_d(q_{k+\alpha}\ \ f_{k+\alpha}) \quad \text{subject to:}$$

$$q_0 = q(0) \quad q_{N_d} = q(T)$$

$$\frac{1}{h}(\partial_v L_0 - p(0)) - (1-\alpha)\partial_q L_0 = (1-\alpha) f_\alpha \tag{10a}$$

$$\begin{aligned} &\frac{1}{h}(\partial_v L_k - \partial_v L_{k-1}) - (1-\alpha)\partial_q L_k - \alpha \partial_q L_{k-1} \\ &= (1-\alpha) f_{k+\alpha} + \alpha f_{k-1+\alpha} \quad \text{for } k = 1 \qquad N_d - 1 \end{aligned} \tag{10b}$$

$$\frac{1}{h}(p(T) - \partial_v L_{N_d-1}) - \alpha \partial_q L_{N_d-1} = \alpha f_{N_d-1+\alpha} \tag{10c}$$

$$f_{min} \leq f_k \leq f_{max} \quad \text{for } k = 0 \qquad N_d \tag{10d}$$

The additional equations (10a) and (10c) correspond to initial and final velocity constraints. In the variational framework such constraints are formulated as initial and final balance of momentum conditions, see [14]. As such, they are written in terms of given initial and final momenta $p(0) = \partial_{\dot{q}} L(q(0)\ \ \dot{q}(0))$ and $p(T) = \partial_{\dot{q}} L(q(T)\ \ \dot{q}(T))$ that can be directly computed from the given velocities $\dot{q}(0)$ and $\dot{q}(T)$.

Example: Consider the case of a linear system with Lagrangian $L(q\ v) = \frac{1}{2}\left(v^T M v - q^T K q\right)$, where M is the system mass matrix and K is a quadratic potential (e.g. from elastic energy) matrix and let $f = u - Cv$, where u is control input and C is a damping matrix. Then, denoting the discrete velocity by

$$v_k = \frac{q_{k+1} - q_k}{h}$$

the equations of motion become:

$$\begin{aligned}
&M\frac{v_0 - v(0)}{h} - (1-\alpha)Kq_0 = (1-\alpha)(u_\alpha - Cv_0)\\
&M\frac{v_k - v_{k-1}}{h} - (1-\alpha)Kq_{k+\alpha} - \alpha K q_{k-1+\alpha}\\
&= (1-\alpha)(u_{k+\alpha} - Cv_k) + \alpha(u_{k-1+\alpha} - Cv_{k-1})\\
&M\frac{v(T) - v_{N_d-1}}{h} - \alpha K q_N = \alpha(u_{N_d-1+\alpha} - Cv_{N_d-1})
\end{aligned}$$

for $k = 1 \quad N_d - 1$. The example of actuator studied in this paper (see, §III) is based on such a model with the simple case $Q = \mathbb{R}$ and appropriately chosen damping and spring coefficients. While simplistic, in the context of this paper, such a model is useful to demonstrate the combined optimal control of the reaction-diffusion parabolic PDE and actuator arm. Our ongoing work considers a more realistic rigid body actuator with velocity constraints that exploit the structural preservation properties of the variational discretization as described in [6].

An example of an optimized trajectory for the system (2) realized by eqs.(10a),10b,10c is given in Fig.2, in which the control-effort minimizing trajectory between the two points starting and ending with zero velocity is provided. The example was computed by formulating a quadratic program in the DMOC implementation.

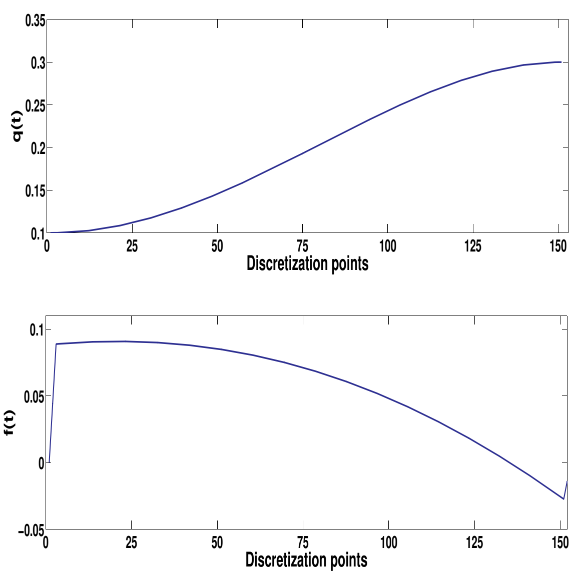

Fig. 2. Solution to the DMOC problem given by eqs.(10a),(10b),(10c) for (2) with the scaled parameters $L = 10$, $K = 2\ L$, $M = 10\ L^2$, $C = 10^{-3}\ L$.

B. Parabolic PDEs

In this work, we consider a class of distributed parameter systems that can be represented by the linear parabolic PDEs of the following form:

$$\frac{\partial x}{\partial t} = \kappa\frac{\partial^2 x}{\partial \zeta^2} + \eta x + b_{(q(t))}u(t) \quad 0 \le \zeta \le l \tag{11}$$

$$y(t) = \int_0^l c_{d_j}(\zeta) x d\zeta \tag{12}$$

with the following boundary and initial conditions:

$$\frac{\partial x}{\partial \zeta}\ {}_{\zeta=0} = 0 = \frac{\partial x}{\partial \zeta}\ {}_{\zeta=1} \tag{13}$$

subject to the following input and state constraints:

$$u^{min} \le u(t) \le u^{max} \tag{14}$$

$$x^{min} \le \int_0^l r_w(\zeta) x(\zeta\ t) dz \le x^{max} \quad w = 1 \cdots g \tag{15}$$

where $x(\zeta\ t)$ denotes the state variable, $\zeta \in [0\ l]$ is the spatial coordinate, $t \in [0\ \infty)$ is the time, $u_{(t)} \in \mathrm{R}$ denotes constrained manipulated input; u^{min} and u^{max} are real numbers representing the lower and upper limits associated with the input applied at the actuator location, and x_w^{min} and x_w^{max} are real numbers representing the lower and upper state constraints enforced at $w-th$ constraints location, $y(t)$ is the output variable obtained by d_j-th sensor. The term $\dfrac{\partial^2 x(\zeta\ t)}{\partial \zeta^2}$ denotes the second-order spatial derivative of $x(\zeta\ t)$ and $x_0(\zeta)$ is a sufficiently smooth function of ζ. The function $c_{d_j}(\zeta) \in L_2(0\ l)$ shows how the sensing is distributed within the spatial interval $[0\ l]$. In (15), the function $r_w(\zeta) \in L_2(0\ l)$ is the "state constraint distribution" function which is a square-integrable and describes how the $w-th$ state constraint is enforced within the spatial domain $[0\ l]$. The state space of interest is $\mathcal{H} = L_2(0\ l)$, with the standard inner product $(\cdot\ \cdot)$ and the norm $\|\cdot\|$ defined on it. The PDE of eqs.(11),-(12) is formulated as an abstract evolutionary equation in the state space $\mathcal{H} = L_2(0\ l)$ as follows:

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_{q(t)}u(t) \qquad x(0) = x_0 \tag{16a}$$

$$y(t) = \mathcal{C}_j x(t) \tag{16b}$$

where the operator $\mathcal{A}$ is defined as:

$$\mathcal{A}\phi = \kappa\frac{d^2\phi}{d\zeta^2} + \eta\phi \qquad 0 < \zeta < l \tag{17}$$

where $\phi(\zeta)$ is a smooth function on $[0\ l]$, with the following dense domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \ &\phi(z) \in L_2(0\ \pi) : \phi(z)\ \frac{d\phi(z)}{dz} \text{ are abs. cont.}\\ &\mathcal{A}\phi \in L_2(0\ l)\ \phi'(0) = 0 = \phi'(l) = 0\end{aligned} \tag{18}$$

the input operator as:

$$\mathcal{B}_{(t)}u(t) = b_{q(t)}(\cdot)u(t) \tag{19}$$

and the output operator as:

$$\mathcal{C}_j x(t) = (c_{d_j}(\cdot)\ x(\zeta\ t)) \tag{20}$$

and the PDE state constraints as:

$$x_w^{min} \leq (r_w(\cdot)\ x(\zeta\ t)) \leq x_w^{max} \tag{21}$$

Using the above definitions, the system of eqs.(11),(13),(12),(14),(15) can be written as the family of switched systems which is parametrized by the actuator and sensing distribution function, which takes the following abstract evolutionary equation form,

$$\begin{aligned}
\dot{x}(t) = &\quad \mathcal{A}x(t) + \mathcal{B}_{q(t)}u(t) \qquad x(0) = x_0 && \text{(22a)}\\
y(t) = &\quad \mathcal{C}_j x(t) \qquad j = 1\ \cdots\ m_s && \text{(22b)}\\
u_\iota^{min} \leq &\quad u(t) \leq u_\iota^{max} \quad \iota = 1\ \cdots\ m_a && \text{(22c)}\\
x_w^{min} \leq &\quad \mathcal{S}_w x(t) \leq x_w^{max} \quad w = 1\ \cdots\ g && \text{(22d)}
\end{aligned}$$

on $\mathcal{H} = L_2(0\ l)$. The spectrum of the Riesz spectral operator $\mathcal{A}$ can be obtained by solving the eigenvalue problem: $\mathcal{A}\phi = \lambda\phi$ with boundary conditions (13). In the case of a self-adjoint operator, the eigenvalues of $\mathcal{A}$ are real, and for a given κ and η, only a finite number of unstable eigenvalues exists, and the distance between any two consecutive eigenvalues (i.e., λ_i and λ_{i+1}) increases as κ increases. This implies that the dominant dynamics of the PDE can be described by a finite–dimensional system, and motivates the use of modal decomposition to derive a finite–dimensional system that well approximates dynamics of the PDE.

Remark 1: It is important to emphasize that the input and output operators $\mathcal{B}_{q(t)}$ and $\mathcal{C}_j$ in (22a)-(22b) are parametrized by the spatial location of the ι-th actuator and j-th sensor, which invokes condition on the generic property of the controllability and observability of the evolutionary equation (22a)-(22b). Namely, in the context of proposed predictive control optimal switching policy with collocated actuators and sensors developed in the ensuing sections, we assume that the set of all available actuator locations considered preserves approximate controllability condition for the $(\mathcal{A}\ \mathcal{B}_{q(t)})$ pair, and approximate observability condition for the $(\mathcal{A}\ \mathcal{C}_j)$ pair [2].

C. Modal decomposition

In this section, we apply standard modal decomposition to the infinite-dimensional system of (22a) to obtain a finite-dimensional system. Let us define a spectral projection operator $\mathcal{P}_s$ which induces the following decomposition of the separable Hilbert space $\mathcal{H}$ into two subspaces $\mathcal{H}_s$ and $\mathcal{H}_f$, $\mathcal{H} = \mathcal{H}_s \bigoplus \mathcal{H}_f$, so that $\mathcal{H}_s = \mathcal{P}_s\mathcal{H}$, and $\mathcal{H}_f = (I - \mathcal{P}_s)\mathcal{H} = \mathcal{P}_f\mathcal{H}$, see [2]. The state $z(t)$ of the system of (22a) can be decomposed as:

$$\begin{aligned}
\dot{x}_s(t) &= \mathcal{A}_s x_s(t) + \mathcal{B}_{s\iota}u(t) \quad x_s(0) = \mathcal{P}_s x(0)\\
\dot{x}_f(t) &= \mathcal{A}_f x_f(t) + \mathcal{B}_{f\iota}u(t) \quad x_f(0) = \mathcal{P}_f z(0) \qquad (23)\\
y(t) &= \mathcal{C}_{sj}x_s(t) + \mathcal{C}_{fj}x_f(t)
\end{aligned}$$

where $\mathcal{A}_s = \mathcal{P}_s\mathcal{A}$, $\mathcal{B}_{s\iota} = \mathcal{P}_s\mathcal{B}_\iota$, $\mathcal{A}_f = \mathcal{P}_f\mathcal{A}$, $\mathcal{B}_{f\iota} = \mathcal{P}_f\mathcal{B}_\iota$, $\mathcal{C}_{sj} = \mathcal{C}_j\mathcal{P}_s$, $\mathcal{C}_{fj} = \mathcal{C}_j\mathcal{P}_f$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = diag\ \lambda_{\nu\kappa}$ ($\lambda_{\nu\kappa}$ are possible unstable eigenvalues of $\mathcal{A}_s$, $\kappa = 1 \quad m$) and $\mathcal{A}_f$ is an infinite dimensional operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$). We consider a high fidelity approximation ($\aleph$-th order approximation) of the $x_s(t)$- and $x_f(t)$- subsystems (23) which can be transformed into an appropriate discrete equivalent of the continuous dynamics, when the ideal sampler is used.

D. Model Predictive Control

In the controller synthesis utilized in this work, a linear time invariant discrete model of the system is considered and it is given in the following form:

$$\begin{aligned}
x(\iota + 1) &= Ax(\iota) + Bu(\iota) \qquad (24)\\
y(\iota) &= Cx(\iota)
\end{aligned}$$

where $x(\iota) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. A stabilizing regulator can be determined as the solution of minimization of the following infinite horizon open-loop quadratic objective function at the time ι:

$$\begin{aligned}
\Phi(\iota) = &\sum_{i=1}^{\infty} x(\iota + i\ \iota)^T C^T Q C x(\iota + i\ \iota)\\
&+ u(\iota + i\ \iota)^T R u(\iota + i\ \iota) +\\
&\triangle u(\iota + i\ \iota)^T P \triangle u(\iota + i\ \iota) \qquad (25)
\end{aligned}$$

where $Q = Q^T \geq 0$, $R = R^T > 0$, $P = P^T \geq 0$, term $\triangle u(\iota + i\ \iota) = u(\iota + i\ \iota) - u(\iota + i - 1\ \iota)$ is the change of the input vector at the time ι, $x(\iota + i\ \iota)$ and $u(\iota + i\ \iota)$ denote the variable $x(\cdot)$ and $u(\cdot)$ at a sampling time $\iota + i$ predicted at the given sampling time ι. At the time $\iota + N$, the control input vector $u(\iota + i\ \iota)$ is set to zero and kept at this value for all $i \geq N$ in the open-loop objective function value calculation, so that at sampling time ι, the control move $u(\iota)$ equals the first element $u(\iota\ \iota)$ of the sequence $[u(\iota\ \iota)\ \cdots\ u(\iota + N - 1\ \iota)]$ which is the minimizer of the optimization problem:

$$\begin{aligned}
J(\iota) = &\min_{u(\iota\ \iota)\ \cdots\ u(\iota+N-1\ \iota)} \Phi(\iota) + \epsilon^T(\iota) Q \epsilon(\iota) \qquad (26)\\
&\triangle u(\iota + i\ \iota) \leq \triangle u^{max} \qquad i = 0\ 1\ \cdots\ N\\
&u^{min} \leq u(\iota + i\ \iota) \leq u^{max}\\
&Gx(\iota + i\ \iota) \leq g + \epsilon(i\ \iota)
\end{aligned}$$

where $G \in \mathbb{R}^{n_G \times n}$, $g \in \mathbb{R}^{n_G}$, and $Q > 0$ is diagonal. Input constraints represent physical limitations on actuators which cannot be violated under any circumstances, while the output/state constraints can be softened by slack variables $\epsilon(\iota)$ and can be temporally violated if necessary. Stability properties of the state feedback controller given by (26) is conditionally connected with the feasibility issue of the constrained optimization problem [15], [16], [9].

III. Actuator guidance and scheduling policy by modal model predictive control

In this section, a novel optimal predictive control law that accounts for the input and state constraints with an optimal actuation/sensing policy calculated by DMOC is considered. It explores the optimality in the sense of the best location utilized within a finite set of available locations with respect to actuation/sensing under the presence of state and input constraints, under the cost of translating the actuator from one prespecified position to another. In particular, we consider the case of moving actuator among prespecified positions within domain, fixed sensor architecture with associated input constraints and PDE state constraints fixed at prespecified locations. Such flexibility of the actuator architecture needs to be allowed either due to the process specifications or the actuator's ability to provide control signal with desirable speed and accuracy [17], [18], [19]. The predictive control law formulation of the movable-actuator fixed-sensing architecture relies on some assumptions, see [18]. Namely, we assume that there is a finite number of admissible prespecified actuator arm locations, denoted by $p_i(\zeta) = \; p_1(\zeta) \; p_2(\zeta) \; \cdots \; p_{m_a}(\zeta) \; \in P$. An important factor in the implementation of a movable actuator fixed sensing architecture is the time required by the actuation device to transverse from one location $p_j(\zeta)$ to $p^*(\zeta)$ location and we assume that the actuator arm is faster than the fastest unstable dynamics in the parabolic PDE model. However, the important attribute of the proposed fuzzed DMOC methodology and model predictive control formulation is the flexible ability to account for the speed of the actuation device to transverse from one location to another through an additional term in the performance functional (25) representing the weight associated with the transfer of the actuation device from the given current location to all other available actuator locations. With this in mind, a combined optimization problem is given as follows:

$$\min_{u(t)\; f(t)} \int_0^T (y(t)'Qy(t) + u(t)'Ru(t)' + f'(t)\bar{Q}f(t))dt \tag{27a}$$

$$M\ddot{q} + C\dot{q} + kq = f(t) \tag{27b}$$

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_{q(t)}u(t) \quad x(0) = x_0 \tag{27c}$$

$$y(t) = \mathcal{C}x(t) \tag{27d}$$

$$u^{min} \leq u(t) \leq u^{max} \tag{27e}$$

$$x_w^{min} \leq \mathcal{S}_w x(t) \leq x_w^{max} \tag{27f}$$

can be reformulated in the discrete DMOC & MMPC setting which is constructed in optimal and realizable (QP) optimization. Namely, just a straightforward discretization of eqs.(27a)–(27f) will lead to nonconvex constrained optimization problem which does not exploit any features of the underlying systems' dynamics representation nor the structure of the optimization.

In order to account in the model modal predictive control formulation for the optimization problem (27a)–(27f), the MPC formulation [15], [9] with DMOC can be taken into the following form:

$$\begin{aligned}\min_u \sum_{i=0}^{N-1} \Big[& x_s(\iota + i \; \iota)^T \tilde{\mathcal{C}}_{sj}^T Q \tilde{\mathcal{C}}_{sj} x_s(\iota + i \; \iota) \\ & + u(\iota + i \; \iota)^T \mathcal{R} u(\iota + i \; \iota) \Big] + \\ & + x_s(\iota + N \; \iota)^T \overline{Q}_{p_j} x_s(\iota + N \; \iota) + \\ & + \min_{q_{0:N_d} \; f_{0:N_d}} \sum_{k=0}^{N_d-1} \Theta_d(q_{k+\alpha} \;\; f_{k+\alpha})\end{aligned} \tag{28}$$

$$x_s(i+1 \; \iota) = \tilde{\mathcal{A}}_s x_s(i \; \iota) + \tilde{\mathcal{B}}_s(q_k)u(i \; \iota) \tag{29a}$$

$$x_f(i+1 \; \iota) = \tilde{\mathcal{A}}_f x_f(i) + \tilde{\mathcal{B}}_f(q_k)u(i \; \iota) \tag{29b}$$

$$u^{min} \leq u(i \; \iota) \leq u^{max} \tag{29c}$$

$$\tilde{\mathcal{S}}_{sw} x_s(i \; \iota) \leq x_w^{max} - \tilde{\mathcal{S}}_{sf} x_f(i \; \iota) \tag{29d}$$

$$-\tilde{\mathcal{S}}_{sw} x_s(i \; \iota) \leq -x_w^{min} + \tilde{\mathcal{S}}_{sf} x_f(i \; \iota) \tag{29e}$$

$$x_{us}(N) = 0 \qquad i = 0 \; 1 \; \cdots \; N-1 \tag{29f}$$

$$M\frac{v_0 - v(0)}{h} - (1-\alpha)Kq_0 = (1-\alpha)(u_\alpha - Cv_0) \tag{29g}$$

$$\begin{aligned} & M\frac{v_k - v_{k-1}}{h} - (1-\alpha)Kq_{k+\alpha} - \alpha K q_{k-1+\alpha} \\ & = (1-\alpha)(u_{k+\alpha} - Cv_k) + \alpha(u_{k-1+\alpha} - Cv_{k-1}) \end{aligned} \tag{29h}$$

$$\begin{aligned} & M\frac{v(T) - v_{N_d-1}}{h} - \alpha K q_{N_d} = \alpha(u_{N_d-1+\alpha} - \\ & Cv_{N_d-1}) \quad k = 1 \qquad N_d - 1 \end{aligned} \tag{29i}$$

$$q_0 = q(0) \quad q_{N_d} = q(T) \quad v(0) = 0 \;\; v(T) = 0 \tag{29j}$$

where index k is associated with the positions of the actuator arm which is provided by the solution of the DMOC problem. In other words, we can decompose eqs.(29a)–(29j) into the solution of the optimal trajectory by DMOC algorithm from the given arm position ($p^*(\zeta)$) to all prespecified arm positions ($p_j(\zeta)$) and obtain associated cost of arm displacements, and for chosen transfer calculate the MMPC problem which is parametrized by the calculated trajectories (that is q_k, $[q_0 \;\; q_1 \;\;\; \cdots q_{N_d}]$) of the chosen actuator arm transfer.

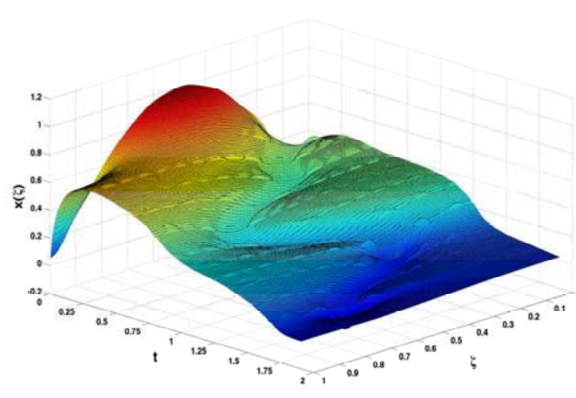

Fig. 3. Stabilization of the state of the parabolic PDE by moving actuator.

Actuator arm activation policy

a In the realization of the actuator arm activation policy, a decision to transfer actuator to a more optimal position with state and input constraints satisfaction, is based on the minimal cost criteria which combines the cost of transfer to the new position and the cost associated with MMPC at that new position without any violation of input and state constraints. In the operational mode, the controller calculates the cost of transfer from the current state to all other prespecified locations through the DMOC setting which yields a discrete mechanics quadratic problem in the configuration q_k and input f_k space. This cost can be easily computed since it is a quadratic problem with linear constraints. The displacement arm cost and the cost of the MMPC controller for the prespecified locations are merged in the decision criteria of moving to another or staying at a given location. Once the minimal cost is found, in the case of moving actuator arm decision, the MPC is subsequently solved at each time instance of the actuator arm movement which is characterized in eqs.(29a)–29e by parametrization of the input operator $\tilde{\mathcal{B}}_f(q_k)$ and $\tilde{\mathcal{B}}_s(q_k)$ through discrete arm positions. In the case of not moving actuator arm decision, the standard MPC controller given by eqs.(29a)–(29e) provides the input to the underlying PDE system. Therefore, the controller activation policy can be formulated as follows:

- **Actuator arm activation policy algorithm**
  (1) At time instance ι and given actuator position p_j and for all m_a-prespecified actuator positions, $p_j = [p_1\ p_2\ \cdots\ p_{m_a}]$, a standard linear predictive control program (29a)–(29e) is constructed. A DMOC problem is constructed as it is given in eqs.(29a),(29f)–(29j) for all possible combinations of actuator positions,
  (2) At the time instance ι, a quadratic constrained predictive control programs for MMPC and DMOC are solved and among the possible actuator moves choose the one that minimizes the cost functional given by:

$$p^* = arg \min_{p_i} [J(x(\iota)\ \ k\ \ u(\cdot)\ \ p_i)+ \\ +J_{dmoc}(q(k)\ \ \dot{q}(k)\ \ f(k)\ \ p_i\ \ p_j)] \qquad (30)$$

  (3) If the smallest cost implies $p_j \neq p^*$, move arm to p^* and solve the family of the model predictive control programs eqs.(29a)–(29e) which are parametrized by the $q(t)$ evolution for each instance of actuator transition from p_j to p^* and applied MPC input as the arm moves in time at each q_k
  (4) Repeat step (1)

Stability property of the aforementioned actuators/sensor activation policy algorithm through model predictive control law eqs.(29a)–(29e) is ensured by asymptotic stabilization over the horizon length of unstable modes of the operator $\tilde{\mathcal{A}}_s$ by the model predictive control algorithm [20]. In this work, we do not analyze closed-loop system properties under the optimal integrated actuator arm activation policy. Comprehensive and thorough analysis of switched distributed parameter systems that can be applied within the context of this work is already given in the work of Demetriou et al [18].

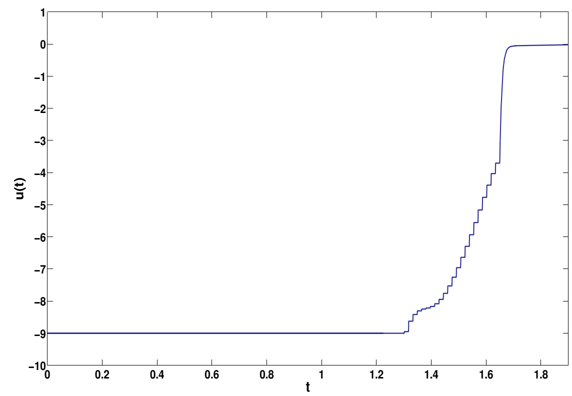

Fig. 4. Constrained input profile evolution.

The aforementioned actuator activation policy within the modal model predictive control framework has four essential merits. First, it exploits particular linear structure of modal representation of the parabolic PDE state in the infinite dimensional setting. Second, it utilizes the best features of the modal model predictive control constructed constraints relaxation optimization algorithm based on the use of the penalty function [15] and it addresses the optimality from the standpoint of the best location for the implementation of an actuator/sensor device. Finally, it uses an easily implementable discrete mechanics optimal control framework, which also preserves mechanical system properties. In this work, only state feedback structure is considered as the control law utilizes the knowledge of entire modal state evolution in its structure. The extension of proposed control law in the case of output feedback realization will heavily rely on the accurate estimation of the modal dynamics, and this issue is not considered in this work.

In the simulations studies, we consider the unstable linear parabolic PDE system with the Neumann boundary conditions, with the following parameters applied in the MMPC controller, weights in MMPC controller are $Q = 100$ and $R = 0\ 01$, 15 eigenfunctions are used to approximate the infinite dimensional system, $[u_{min}\ u_{max}] = [-9\ 9]$, $[x_{min}\ x_{max}] = [-1\ 2]$, there are five prespecified actuator positions $p_j = [0\ 1\ 0\ 3\ 0\ 5\ 0\ 7\ 0\ 9]$, and the state constraint function is given as $r_w(\zeta) = \delta(\zeta - 0\ 41)$, the initial condition $x(0) = sin(\pi\zeta)$ and the horizon length is $N = 50$, $\kappa = 0\ 15$ and $\eta = 1\ 6631$, while DMOC parameters are $N_d = 25$ and $h = 0\ 016$. In Fig.3, the arm is initiated at $p_1 = 0\ 1$ and moves to another position while the stabilization is ensured together with input constraints satisfaction, see Fig.4. Associated with this figure is Fig.5 which describes the optimal arm evolution within domain.

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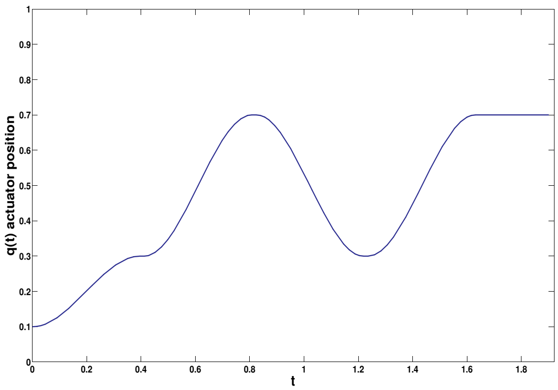

Fig. 5. Optimal arm evolution within domain.

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