

EN530.678 Nonlinear Control and Planning in Robotics

Lecture 9: Backstepping

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1 Backstepping

Backstepping is a nonlinear control design tool for underactuated systems. Backstepping is used for output stabilization or tracking using feedback similarly to feedback linearization. In feedback linearization the stabilizing controller is designed entirely for a virtual input which is then mapped back to the physical input by completely canceling nonlinearities. In backstepping the system is stabilized in stages by “closing the loop” recursively which enables nonlinearities to be exploited, e.g. if natural damping was present in the system. In addition, backstepping becomes more robust to uncertainties.

Example 1. *Stabilization for the unicycle.* Consider the unicycle model with state position (x_1, x_2) and orientation x_3

$$\begin{aligned}\dot{x}_1 &= \cos x_3 u_1 \\ \dot{x}_2 &= \sin x_3 u_1 \\ \dot{x}_3 &= u_2\end{aligned}\tag{1}$$

The task is to stabilize the position, defined by the output $y = (x_1, x_2)$, to the origin $(0, 0)$. Start by defining the Lyapunov function

$$V_0(y) = \frac{1}{2} y^T y$$

Differentiating we obtain

$$\dot{V}_0(y) = y^T \begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix},\tag{2}$$

Our goal is to make \dot{V}_0 negative which will hold if we choose the controls so that

$$\begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix} = -k_0 y, \quad k_0 > 0$$

In general, no control u_1 can be chosen to satisfy this directly. Rewrite (2) according to

$$\dot{V}_0(y) = y^T [-k_0 y + z],$$

where

$$z = k_0 y + \begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix}.$$

The term z defines an error variable that we want to drive to zero which motivates the new Lyapunov function

$$V(y, z) = V_0(y) + \frac{1}{2}z^T z.$$

Differentiating, we obtain (after some manipulation)

$$\dot{V} = -k_0 y^T y + z^T \left[y + k_0 \dot{y} + R(x_3) \begin{pmatrix} \dot{u}_1 \\ u_1 u_2 \end{pmatrix} \right],$$

where

$$R(x_3) = \begin{pmatrix} \cos x_3 & -\sin x_3 \\ \sin x_3 & \cos x_3 \end{pmatrix}.$$

Now, we can set

$$\begin{pmatrix} \dot{u}_1 \\ u_1 u_2 \end{pmatrix} = R(x_3)^T (-y - k_0 \dot{y} - kz), \quad k > 0$$

which results in

$$\dot{V} = -k_0 y^T y - kz^T z \leq 0,$$

and hence we have proved that (y, z) asymptotically stabilizes to $(0, 0)$. The actual controls are

$$\begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{u_1} \end{bmatrix} R(x_3)^T (-y - k_0 \dot{y} - kz). \quad (3)$$

The relationship (3) is used to obtain \dot{u}_1 which is integrated to obtain u_1 which is used to control the system. This is possible as long as $u_1 \neq 0$ which is a structural condition for nonholonomic systems indicating that the vehicle must always maintain non-zero velocity during stabilization. It can be shown that this condition does not pose practical problems since it occurs only infinitely close to the origin [?].

Example 2. *Tracking for the unicycle.* It is often preferable to not directly try to stabilize to given state x_f (especially if it is far away) but instead to first design a good (i.e. optimized) reference trajectory $x_d(t)$ that reaches x_f at some time t_f . This reference trajectory can then be simply tracked until x_f is reached. Consider tracking a unicycle trajectory defined through its position $y(t) = (x_1(t), x_2(t))$, i.e. so that the system should asymptotically track a given reference $y_d(t)$. The key difference from Example 1 is that we would like to asymptotically bring the error

$$e(t) = y(t) - y_d(t),$$

to zero, and so the control law and associated Lyapunov function will be time-dependent:

$$V_0(t, e) = \frac{1}{2}e(t)^T e(t) \quad (4)$$

Differentiating we obtain

$$\dot{V}_0(t, e) = e(t)^T \left[\begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix} - \dot{y}_d(t) \right], \quad (5)$$

Our goal is to make \dot{V}_0 negative which will hold if we choose the controls so that

$$\begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix} = -k_0 e(t) + \dot{y}_d(t), \quad k_0 > 0 \quad (6)$$

In general, no control u_1 can be chosen to satisfy this directly. Rewrite (5) according to

$$\dot{V}_0(t, e) = e^T [-k_0 e + z],$$

where

$$z = -\dot{y}_d(t) + k_0 e + \begin{pmatrix} \cos x_3 u_1 \\ \sin x_3 u_1 \end{pmatrix}.$$

The term z defines an error variable that we want to drive to zero which motivates the new Lyapunov function

$$V(t, e, z) = V_0(t, e) + \frac{1}{2} z^T z.$$

Differentiating, we obtain (after some manipulation)

$$\dot{V} = -k_0 e^T e + z^T \left[e - \ddot{y}_d + k_0 \dot{e} + R(x_3) \begin{pmatrix} \dot{u}_1 \\ u_1 u_2 \end{pmatrix} \right],$$

where

$$R(x_3) = \begin{pmatrix} \cos x_3 & -\sin x_3 \\ \sin x_3 & \cos x_3 \end{pmatrix}.$$

Now, we can set

$$\begin{pmatrix} \dot{u}_1 \\ u_1 u_2 \end{pmatrix} = R(x_3)^T (\ddot{y}_d - e - k_0 \dot{e} - kz), \quad k > 0$$

which results in

$$\dot{V} = -k_0 e^T e - kz^T z \leq 0.$$

The actual controls are

$$\begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{u_1} \end{bmatrix} R(x_3)^T (\ddot{y}_d - e - k_0 \dot{e} - kz). \quad (7)$$

The relationship (7) is used to obtain \dot{u}_1 which is integrated to obtain u_1 which is used to control the system.

Note that we cannot directly conclude asymptotic stability since the system is time-dependent (Lyapunov's direct method as well as LaSalle's principle apply to time-invariant systems). But under the assumption that both $e(t)$ and $\dot{e}(t)$ are bounded, it is possible to apply *Lyapunov exponential stability theorem* (which holds for time-dependent ODEs), and show that the chosen control law results in exponential tracking of the reference $y_d(t)$.

1.1 Integrator Backstepping

Now let's consider the system [?]

$$\dot{\eta} = f(\eta) + g(\eta)\xi \quad (8)$$

$$\dot{\xi} = u \quad (9)$$

where $[\eta^T, \xi] \in \mathbb{R}^{n+1}$ and $u \in \mathbb{R}$ is the control input. The functions $f, g : D \rightarrow \mathbb{R}^n$ are smooth in a domain $D \subset \mathbb{R}^n$ that contains $\eta = 0$ and $f(0) = 0$. The goal is to design a controller which stabilizes the origin ($\eta = 0, \xi = 0$).

Assume that there is a control law $\xi = \phi(\eta)$ which asymptotically stabilizes the subsystem (8) with $\phi(0) = 0$ with associated Lyapunov function $V_0(\eta)$ such that

$$\frac{\partial V_0}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \forall \eta \in D, \quad (10)$$

where $W(\eta)$ is positive definite. Eq (8) can be rewritten as

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)] \\ \dot{\xi} &= u \end{aligned}$$

By defining the error variable

$$z = \xi - \phi(\eta)$$

we have

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= u - \dot{\phi} \end{aligned}$$

Now defining the function

$$V(\eta, \xi) = V_0(\eta) + \frac{1}{2}z^2,$$

as a Lyapunov function candidate, we obtain

$$\begin{aligned} \dot{V} &= \frac{\partial V_0}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V_0}{\partial \eta} g(\eta)z + z\dot{z} \\ &\leq -W(\eta) + \frac{\partial V_0}{\partial \eta} g(\eta)z + z\dot{z} \end{aligned}$$

If we set

$$\dot{z} = -\frac{\partial V_0}{\partial \eta} g(\eta) - kz, \quad k > 0,$$

which is equivalent to

$$u = \frac{\partial \phi}{\partial \eta} [f(\eta) + g(\eta)\xi] - \frac{\partial V_0}{\partial \eta} g(\eta) - k[\xi - \phi(\eta)], \quad (11)$$

and we have

$$\dot{V} \leq -W(\eta) - kz^2.$$

This shows that the origin ($\eta = 0, z = 0$) is asymptotically stable. Since by assumption $\phi(0) = 0$, we conclude that the origin ($\eta = 0, \xi = 0$) is asymptotically stable.

Lemma 1. Consider the system (8)-(9). Let $\phi(\eta)$ be a stabilizing state feedback control law for (8) with $\phi(0) = 0$, and $V(\eta)$ be a Lyapunov function that satisfies (14) with some positive definite function $W(\eta)$. Then, the state feedback control law (11) stabilizes the origin of (8)-(9), with $V(\eta) + [\xi - \phi(\eta)]^2$ as a Lyapunov function. Moreover, if all the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally asymptotically stable.

Example 3. Exploiting nonlinear damping. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u\end{aligned}$$

which can be put into the standard form (8)-(9) with $\eta = x_1$ and $\xi = x_2$. First, we regard x_2 as input that asymptotically stabilizes x_1 . This is accomplished by setting

$$x_2 = \phi(x_1) = -x_1^2 - x_1,$$

which results in the dynamics

$$\dot{x}_1 = -x_1 - x_1^3$$

The Lyapunov function $V_0(x_1) = \frac{1}{2}x_1^2$ satisfies

$$\dot{V}_0 = -x_1^2 - x_1^4 \leq -x_1^2, \quad \forall x_1 \in \mathbb{R}$$

Using Lemma 1 the control

$$\begin{aligned}u &= \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial V_0}{\partial x_1} - [x_2 - \phi(x_1)] \\ &= -(2x_1 + 1)(x_1^2 - x_1^3 + x_2) - x_1 - (x_2 + x_1^2 + x_1)\end{aligned}$$

stabilizes the origin $x = 0$ globally, with a composite Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1)^2$$

1.2 Block Backstepping

Now let's consider the system [?]

$$\dot{\eta} = f(\eta) + G(\eta)\xi \tag{12}$$

$$\dot{\xi} = f_a(\eta, \xi) + G_a(\eta, \xi)u \tag{13}$$

where $\eta \in \mathbb{R}^n, \xi \in \mathbb{R}^m$ and $u \in \mathbb{R}^m$ is the control input. The functions f, f_a, G, G_a are smooth in a domain of interest, f and f_a vanish at the origin and the $m \times m$ matrix G_a is non-singular. Assume that there is a control law $\xi = \phi(\eta)$ which asymptotically stabilizes the subsystem (12) with $\phi(0) = 0$ with associated Lyapunov function V_0 such that

$$\frac{\partial V_0}{\partial \eta}[f(\eta) + G(\eta)\phi(\eta)] \leq -W(\eta), \forall \eta \in D, \tag{14}$$

where $W(\eta)$ is positive definite. Using

$$V = V_0(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^T[\xi - \phi(\eta)]$$

as a Lyapunov function candidate for the overall system we obtain

$$\dot{V} = \frac{\partial V_0}{\partial \eta}(f + G\phi) + \frac{\partial V_0}{\partial \eta}G(\xi - \phi) + [\xi - \phi]^T \left[f_a + G_a u - \frac{\partial \phi}{\partial \eta}(f + G\xi) \right]$$

Taking

$$u = G_a^{-1} \left[\frac{\partial \phi}{\partial \eta}(f + G\xi) - \left(\frac{\partial V_0}{\partial \eta}G \right)^T - f_a - k(\xi - \phi) \right], \quad k > 0$$

results in

$$\dot{V} = \frac{\partial V_0}{\partial \eta}(f + G\phi) - k[\xi - \phi(\eta)]^T[\xi - \phi(\eta)] \leq -W(\eta) - k[\xi - \phi(\eta)]^T[\xi - \phi(\eta)]$$

which shows that the origin ($\eta = 0, \xi = 0$) is asymptotically stable.

Example 4. *Point-mass vehicle.* Consider a point-mass system with position $q \in \mathbb{R}^n$ where $n = 2$ (the case $n = 3$ is identical). The vehicle has mass m and its dynamics are given by

$$m\ddot{q} + b\dot{q} + g = u, \quad (15)$$

where $g = (0, -9.81m)$ denotes gravity, and $b > 0$ is a drag constant. The task is to track a given desired position $q_d(t)$. Using the notation

$$e(t) = q(t) - q_d(t).$$

define the function,

$$V_0(t, e) = \frac{1}{2}\|e(t)\|^2$$

Differentiating, we obtain

$$\dot{V}_0 = e^T \dot{e},$$

which can be made negative by $\dot{e} = -k_0 e$. Re-express V_0 according to

$$\dot{V}_0(t, e) = e^T(-k_0 e + z), \quad k_0 > 0$$

where $z = k_0 e + \dot{e}$ is our new error variable to be driven to zero. Define the new Lyapunov candidate

$$V(t, e, z) = V_0(t, e) + \frac{1}{2}z^T z.$$

Differentiating we obtain

$$\dot{V} = -k_0 e^T e + z^T \left[e - \ddot{q}_d + k_0 \dot{e} + \frac{1}{m}(u - b\dot{q} - g) \right]$$

Choosing

$$u = m(\ddot{q}_d - \dot{e} - k_0\dot{e} - kz) + b\dot{q} + g$$

we have

$$\dot{V} = -k_0e^T e - kz^T z \leq 0.$$

Note that for the point stabilization case (i.e. when $\dot{q}_d = \ddot{q}_d = 0$) we could have just used the block back-stepping to show asymptotic stability. Also note that in this case, the damping term $b\dot{q}$ need not be completely canceled since it can contribute to the control law.

For the general trajectory tracking case, we can again employ Lyapunov's exponential stability theorem to show exponential tracking, assuming $q_d(t)$ and $\dot{q}_d(t)$ are bounded.