

EN530.678 Nonlinear Control and Planning in Robotics

Lecture 8: Feedback Linearization

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1 Introduction

Consider the nonlinear control system

$$\dot{x} = f(x) + G(x)u, \quad (1)$$

$$y = h(x), \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ —the inputs, and $y \in \mathbb{R}^{p \leq m}$ —the output that one wants to achieve, i.e. a trajectory to be tracked.

We are interested in transformation of the form

$$u = a(x) + B(x)v, \quad (3)$$

where $B(x)$ is a nonsingular matrix, and $v \in \mathbb{R}^p$ is a transformed or *virtual input*, so that the input-output response between v and y is linear. This requirement is called *input-output linearization* because only the input-output map h is made linear. On the other hand, *full state linearization* refers to the case when the whole state x must be “linearized”.

Linearization of the form (3) is called *static feedback*.

In some cases one could accomplish the linearization only with the help of extra variables which evolve dynamically. This is the case of *dynamic feedback* which takes the form

$$u = a(x, \xi) + B(x, \xi)v, \quad (4)$$

$$\dot{\xi} = c(x, \xi) + D(x, \xi)v, \quad (5)$$

where ξ are the extra “compensating” variables, also referred to as the *compensator state*. The term *dynamic* is appropriate since ξ has its own additional dynamics that affect the control law.

Feedback linearization, in its simplest form, proceeds by differentiating the output mapping $y = h(x)$ enough times so that all controls appear in a linear, nonsingular relationship to the outputs (or their higher derivatives).

2 Examples from robotics

2.1 Static Feedback

2.1.1 Fully-actuated manipulator control: *computed torque law*

A fully-actuated mechanical system can generally be written as

$$M(q)\ddot{q} + b(q, \dot{q}) = u, \quad (6)$$

where q is the configuration (e.g. the joint angles) and the state is $x = (q, \dot{q})$, with $M(q)$ denoting the mass matrix and $b(q, \dot{q})$ the bias consisting of Coriolis/centrifugal, gravitational and friction terms. Assume that one is interested in tracking a desired path $q_d(t)$. The task is specified by the output

$$y = q.$$

and a virtual input v to be determined, which satisfies

$$\ddot{q} = v,$$

so that

$$u = M(q)v + b(q, \dot{q}) \quad (7)$$

The linear input-output transformation corresponding to (3) is accomplished by setting

$$a(x) = b(q, \dot{q}), \quad (8)$$

$$B(x) = M(q). \quad (9)$$

The control is set to

$$v = \ddot{q}_d - k_d(\dot{q} - \dot{q}_d) - k_p(q - q_d), \quad (10)$$

for some positive scalars $k_d, k_p > 0$. In order to see that the resulting dynamics is linear and stable define the error state $z \in \mathbb{R}^{n=2m}$ according to

$$z = \begin{pmatrix} q - q_d \\ \dot{q} - \dot{q}_d \end{pmatrix}$$

The control law (10) results in the closed-loop linear dynamics

$$\dot{z} = Az,$$

where

$$A = \begin{pmatrix} 0 & I \\ -k_p I & -k_d I \end{pmatrix}$$

is a Hurwitz matrix. The virtual controls are mapped back to the original input u using (7). In general, whenever $\dim(Q) = \dim(U)$ one can always use nonlinear static feedback to achieve linearization.

2.1.2 Partial feedback linearization

Now consider the case of an underactuated system in the form

$$M(q)\ddot{q} + b(q, \dot{q}) = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad (11)$$

where $u \in \mathbb{R}^m$. It can be equivalently expressed as

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + b_1 = 0 \quad (12)$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + b_2 = u, \quad (13)$$

where

$$q_1 \in \mathbb{R}^{\ell=n-m}, \quad q_2 \in \mathbb{R}^m,$$

the mass matrices are defined so that

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

and $b_1(q, \dot{q}) \in \mathbb{R}^\ell$ and $b_2(q, \dot{q}) \in \mathbb{R}^m$ correspond to the bias terms.

Collocated Input/Output Linearization. Assume that one is interested in controlling the output

$$y = q_2 \in \mathbb{R}^m,$$

i.e. the coordinates corresponding to the controlled degrees of freedom. Equations (12)-(13) can be combined to obtain

$$\bar{M}_{22}\ddot{q}_2 + \bar{b}_2 = u, \quad (14)$$

where

$$\bar{M}_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}, \quad (15)$$

$$\bar{b}_2 = b_2 - M_{21}M_{11}^{-1}b_1, \quad (16)$$

$$(17)$$

Define the virtual control $v \in \mathbb{R}^m$ to be determined so that the dynamics $\ddot{q}_2 = v$ solves the tracking problem. From (14) we have

$$u = \bar{M}_{22}v + \bar{b}_2 \quad (18)$$

The control law is set to

$$v = \ddot{y}_d - k_d(\dot{y} - \dot{y}_d) - k_p(y - y_d), \quad (19)$$

In order to see that the resulting dynamics is linear define the error state $z \in \mathbb{R}^{2n}$ according to

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y - y_d \\ \dot{y} - \dot{y}_d \end{pmatrix}$$

The control law (19) results in the closed-loop linear dynamics

$$\dot{z} = Az,$$

where

$$A = \begin{pmatrix} 0 & I_m \\ -k_p I_m & -k_d I_m \end{pmatrix}$$

is a Hurwitz matrix. The virtual controls are mapped back to the original input u using (18). In order to study the evolution of the remaining “non-linearized” coordinates q_1 define the state $\eta \in \mathbb{R}^{2m}$ by

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ \dot{q}_1 \end{pmatrix}.$$

Using (12) we have

$$\dot{\eta}_1 = \eta_2 \tag{20}$$

$$\dot{\eta}_2 = -M_{11}^{-1} (M_{12}(\ddot{y}_d - k_p z_1 - k_d z_2) + b_1). \tag{21}$$

The complete system can be written as

$$\dot{z} = Az \tag{22} \quad : \text{ linearized}$$

$$\dot{\eta} = w(z, \eta, t). \tag{23} \quad : \text{ non-linearized.}$$

The *zero dynamics* of the system is then defined as the evolution of the non-linearized part after the linear part has stabilized, i.e. at $z = 0$ or

$$\dot{\eta} = w(0, \eta, t). \tag{24}$$

Theorem 1. (see [?]) Consider the system (22)-(23). Suppose that $w(0, \eta_0, t) = 0$ for $t \geq 0$, i.e. $(0, \eta_0)$ is an equilibrium of the full system(22)-(23) and η_0 is an equilibrium of the zero dynamics (24). Suppose also that A is a Hurwitz matrix. Then $(0, \eta_0)$ is locally stable (respectively, locally asymptotically stable, unstable) if η_0 is locally stable (respectively, locally asymptotically stable, unstable) for the zero dynamics (24).

Non-located Input/Output Linearization. Assume that one is interested in achieving the output

$$y = q_1 \in \mathbb{R}^\ell,$$

i.e. the coordinates corresponding to the unactuated degrees of freedom. This would only be possible if $\ell \leq m$, i.e. if the number of controlled DOF is at least as large as the uncontrolled DOF. In addition, the system must be *strongly inertially coupled*, i.e. more formally it must satisfy

$$\text{rank}(M_{12}) = \ell.$$

If the system is strongly coupled, then equations (12)-(13) can be combined to obtain

$$\tilde{M}_{21} \ddot{q}_1 + \tilde{b}_2 = u, \tag{25}$$

where

$$\tilde{M}_{21} = M_{21} - M_{22} M_{12}^\dagger M_{11}, \tag{26}$$

$$\tilde{b}_2 = b_2 - M_{22} M_{12}^\dagger b_1, \tag{27}$$

$$\tag{28}$$

where the right *pseudo-inverse* M_{12}^\dagger of a matrix M_{12} is defined by

$$M_{12}^\dagger = M_{12}^T (M_{12} M_{12}^T)^{-1}$$

Notice that $\text{rank}(M_{12}) = \ell \Rightarrow \text{rank}(M_{12}M_{12}^T) = \ell$ and the inversion is valid since $M_{12}M_{12}^T \in \mathbb{R}^{\ell \times \ell}$.

Define the virtual control $v \in \mathbb{R}^\ell$ to be determined so that the dynamics $\ddot{q}_1 = v$ solves the tracking problem. From (25) we have

$$u = \tilde{M}_{21}v + \tilde{b}_2. \quad (29)$$

The control law is set to

$$v = \ddot{y}_d - k_d(\dot{y} - \dot{y}_d) - k_p(y - y_d), \quad (30)$$

In order to see that the resulting dynamics is linear define the error state $z \in \mathbb{R}^{2\ell}$ according to

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y - y_d \\ \dot{y} - \dot{y}_d \end{pmatrix}$$

The control law (10) results in the closed-loop linear dynamics

$$\dot{z} = Az,$$

where

$$A = \begin{pmatrix} 0 & I_\ell \\ -k_p I_\ell & -k_d I_\ell \end{pmatrix}$$

is a Hurwitz matrix. The virtual controls are mapped back to the original input u using (29). In order to study the evolution of the remaining “non-linearized” coordinates q_1 define the state $\eta \in \mathbb{R}^{2m}$ by

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ \dot{q}_2 \end{pmatrix}.$$

Using (12) we have

$$\dot{\eta}_1 = \eta_2 \quad (31)$$

$$\dot{\eta}_2 = -M_{12}^\dagger (M_{11}(\ddot{y}_d - k_p z_1 - k_d z_2) + b_1). \quad (32)$$

The complete system can be written as

$$\dot{z} = Az \quad : \text{ linearized} \quad (33)$$

$$\dot{\eta} = w(z, \eta, t). \quad : \text{ non-linearized.} \quad (34)$$

The *zero dynamics* of the system is then defined as the evolution of the non-linearized part after the linear part has stabilized, i.e. at $z = 0$ or

$$\dot{\eta} = w(0, \eta, t). \quad (35)$$

Theorem 1 applies to this case as well.

Example 1. *The two-link robot. The equations of motion of a two-link robot are*

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + b_1 = u_1 \quad (36)$$

$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + b_2 = u_2, \quad (37)$$

where

$$\begin{aligned}
m_{11} &= m_1 \ell_{c1}^2 + m_2 (\ell_1^2 + \ell_{c2}^2 + 2\ell_1 \ell_{c2} \cos q_2) + I_1 + I_2 \\
m_{22} &= m_2 \ell_{c2}^2 + I_2, \\
m_{12} &= m_{21} = m_2 (\ell_{c2}^2 + \ell_1 \ell_{c2} \cos q_2) + I_2 \\
b_1 &= h_1 + \phi_1, \quad b_2 = h_2 + \phi_2 \\
h_1 &= -m_2 \ell_1 \ell_{c2} \sin q_2 \dot{q}_2^2 - 2m_2 \ell_1 \ell_{c2} \sin(q_2) \dot{q}_2 \dot{q}_1 \\
h_2 &= m_2 \ell_1 \ell_{2c} \sin(q_2) \dot{q}_1^2 \\
\phi_1 &= (m_1 \ell_{c1} + m_2 \ell_1) g \cos(q_1) + m_2 \ell_{c2} g \cos(q_1 + q_2) \\
\phi_2 &= m_2 \ell_{c2} g \cos(q_1 + q_2),
\end{aligned}$$

where we split b_i into the Coriolis terms h_i and gravity terms ϕ_i for $i = 1, 2$, with $g = 9.8m/s^2$. Setting $u_1 = 0$ corresponds to the Acrobot while $u_2 = 0$ the Pendubot. Another example with $\phi_1 = \phi_2 = 0$ and $u_2 = 0$ is the underactuated manipulator.

Example 2. *Cart-pole system.* Even-though this is a classical academic example it still has interesting challenges from the standpoint of nonlinear control [?]. The dynamics are given by:

$$(m_p + m_c)\ddot{x} + m_p \ell \cos \theta \ddot{\theta} - m_c \dot{\theta}^2 \sin \theta = F, \quad (38)$$

$$m_p \ell \cos \theta \ddot{x} + m_p \ddot{\theta} - m_p \ell g \sin \theta = 0 \quad (39)$$

The system can be put into standard form by setting $q_1 = \theta$, $q_2 = x$, $u = F$ and after normalizing the constants

$$\ddot{q}_1 + \cos q_1 \ddot{q}_2 - \sin q_1 = 0, \quad (40)$$

$$\cos q_1 \ddot{q}_1 + 2\ddot{q}_2 - \dot{q}_1^2 \sin q_1 = u \quad (41)$$

2.2 Dynamic Feedback

2.2.1 Wheeled robot trajectory tracking

Consider the kinematic unicycle model with configuration $q \in \mathbb{R}^3$ denoting its position and orientation. The state of the vehicle is $x = q$ so that (x_1, x_2) denote the position and x_3 the orientation.

Note: we normally use coordinates (x, y, θ) for this example but in order to illustrate the general theory we are using the equivalent $x = (x_1, x_2, x_3)$.

The equations of motion are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2. \quad (42)$$

The task is to track given desired trajectory specified by the positions $(x_{1d}(t), x_{2d}(t))$.

Feedforward Control It is interesting to note that for wheeled vehicles eventhough we only specify the desired position, the orientation is also implicitly specified. This is expressed by the relationship

$$x_{3d} = a \tan 2(\dot{x}_{2d}, \dot{x}_{1d}) + k\pi, \quad k = 0, 1,$$

depending on whether the vehicle is moving forward or backwards.

A feedforward control law then becomes (recall $\frac{d}{dt} \tan^{-1} a = \frac{1}{a^2+1}$)

$$\begin{aligned} u_{1d} &= \pm \sqrt{\dot{x}_{1d}^2 + \dot{x}_{2d}^2}, \\ u_{2d} = \dot{x}_{3d} &= \frac{\dot{x}_{1d}\ddot{x}_{2d} - \ddot{x}_{1d}\dot{x}_{2d}}{\dot{x}_{1d}^2 + \dot{x}_{2d}^2}. \end{aligned}$$

In a perfect (zero-noise) setting if $x(0) = x_d(0)$ then this control law will exactly follow the trajectory.

Feedback Control Tracking with feedback is much more desirable in order to have robustness. Assume that we can measure the position (x_1, x_2) and we would like to track it. We specify that by the output

$$y = h(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is not possible to directly track it since we have only one input u_1 with direct influence on position velocities.

Differentiate the equations of motion (42) to obtain

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\sin x_3 \dot{x}_3 u_1 + \cos x_3 \dot{u}_1 \\ \cos x_3 \dot{x}_3 u_1 + \sin x_3 \dot{u}_1 \end{pmatrix} = \begin{pmatrix} \cos x_3 & -\sin x_3 u_1 \\ \sin x_3 & \cos x_3 u_1 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix}. \quad (43)$$

Eq. (43) means that the accelerations (\ddot{x}_1, \ddot{x}_2) can be specified exactly by choosing (\dot{u}_1, u_2) as long as $u_1 \neq 0$. This suggests we should be thinking of controlling the system through (\dot{u}_1, u_2) or equivalently, as long as the condition $u_1 \neq 0$ holds, through (\ddot{x}_1, \ddot{x}_2) .

More formally, in the language of feedback linearization define the *virtual input* $v \in \mathbb{R}^2$ so that

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = v \quad (44)$$

and the *dynamic compensator* $\xi \in \mathbb{R}$ defined by

$$\xi = u_1.$$

Equation (43) can be solved for $(\dot{\xi}, u_2)$ to obtain

$$\dot{\xi} = v_1 \cos x_3 + v_2 \sin x_3, \quad (45)$$

$$u_2 = \frac{v_2 \cos x_3 - v_1 \sin x_3}{\xi}. \quad (46)$$

The control law is then chosen according to

$$v = \ddot{y}_d - k_p(y - y_d) - k_d(\dot{y} - \dot{y}_d), \quad (47)$$

where k_p and k_d are 2×2 positive definite matrices, to asymptotically track $y_d(t) = (x_{1d}(t), x_{2d}(t))$. In order to see that the resulting dynamics is linear and stable define the error state $z \in \mathbb{R}^4$ according to

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y - y_d \\ \dot{y} - \dot{y}_d \end{pmatrix}$$

The control law (10) results in the closed-loop linear dynamics

$$\dot{z} = Az,$$

where

$$A = \begin{pmatrix} 0 & I \\ -k_p & -k_d \end{pmatrix}$$

is a Hurwitz matrix.

The actual inputs u are computed using (45) and integrating \dot{u}_1 . Note that the control law is valid only when $u_1 \neq 0$ (see [?] for precise treatment of the singularity related to u_1).

The feedback linearized form can be summarized according to:

Definition	General	Wheeled Robot
Outputs:	$y = h(x)$	$y = (x_1, x_2)$
Controls transformation:	$u = a(x, \xi) + B(x, \xi)v$	$u = \begin{pmatrix} \xi \\ v_2 \cos \theta - v_1 \sin \theta \\ \xi \end{pmatrix}$
Dynamic Compensator:	$\dot{\xi} = c(x, \xi) + D(x, \xi)v$	$\dot{\xi} = v_1 \cos \theta + v_2 \sin \theta$

2.2.2 UAV trajectory tracking

Consider a 2D model of a UAV with configuration $q \in \mathbb{R}^3$ where (q_1, q_2) denote its horizontal and vertical positions and q_3 denotes its orientation. The vehicle has mass m and moment of inertia J . The state of the vehicle is $x = (q, \dot{q})$ and evolves according to the dynamics

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = R(x_3) \begin{pmatrix} 0 \\ u_1 \end{pmatrix} + g, \quad (48)$$

$$J\ddot{x}_3 = u_2, \quad (49)$$

where $g = (0, -9.81m)$ denotes gravity, and $R(x_3)$ rotates from body to spatial frame, i.e.

$$R(x_3) = \begin{pmatrix} \cos x_3 & -\sin x_3 \\ \sin x_3 & \cos x_3 \end{pmatrix}. \quad (50)$$

The task is to track a given desired position $(x_{1d}(t), x_{2d}(t))$. It is not possible to control directly the position so we differentiate (48) to get

$$m \begin{pmatrix} \dot{x}_1^{(3)} \\ \dot{x}_2^{(3)} \end{pmatrix} = R(x_3) \begin{pmatrix} -u_1 \dot{x}_3 \\ \dot{u}_1 \end{pmatrix}, \quad (51)$$

where we used the relationship

$$\dot{R}(x_3) = R(x_3) \begin{pmatrix} 0 & -\dot{x}_3 \\ \dot{x}_3 & 0 \end{pmatrix}. \quad (52)$$

The controls (or their derivatives) still do not appear linearly, so repeat differentiation to obtain

$$m \begin{pmatrix} x_1^{(4)} \\ x_2^{(4)} \end{pmatrix} = R(x_3) \left(\begin{pmatrix} -\dot{u}_1 \dot{x}_3 \\ -u_1 \dot{x}_3^2 \end{pmatrix} + \begin{pmatrix} -\dot{u}_1 \dot{x}_3 - u_1 \ddot{x}_3 \\ \ddot{u}_1 \end{pmatrix} \right), \quad (53)$$

or equivalently

$$\begin{pmatrix} x_1^{(4)} \\ x_2^{(4)} \end{pmatrix} = \frac{1}{m} R(x_3) \begin{pmatrix} -2\dot{u}_1 \dot{x}_3 \\ -u_1 \dot{x}_3^2 \end{pmatrix} + \frac{1}{m} R(x_3) \begin{pmatrix} -u_1/J & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_2 \\ \ddot{u}_1 \end{pmatrix}. \quad (54)$$

Similarly to the wheeled robot we can define the *virtual input* $v \in \mathbb{R}^2$ which will be determined so that

$$\begin{pmatrix} x_1^{(4)} \\ x_2^{(4)} \end{pmatrix} = v \quad (55)$$

renders the closed-loop dynamics stable. The presence of time-derivative terms requires the *dynamic compensator* $\xi \in \mathbb{R}^2$ defined by

$$\xi = \begin{pmatrix} u_1 \\ \dot{u}_1 \end{pmatrix}.$$

Equation (54) can be solved for (\ddot{u}_1, u_2) to obtain

$$\begin{pmatrix} u_2 \\ \ddot{u}_1 \end{pmatrix} = \begin{pmatrix} -J/\xi_1 & 0 \\ 0 & 1 \end{pmatrix} \left(mR^T(x_3)v - \begin{pmatrix} -2\xi_2 x_6 \\ -u_1 x_6^2 \end{pmatrix} \right) \quad (56)$$

Then we can set

$$v = y_d^{(4)} - \sum_{i=0}^3 K_i (y^{(i)} - y_d^{(i)}), \quad (57)$$

where K_i are 2×2 positive definite matrices that will be chosen to asymptotically track $y_d(t) = (x_{1d}(t), x_{2d}(t))$. In order to see that the resulting dynamics is linear and stable define the error state $z \in \mathbb{R}^8$ according to

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} y - y_d \\ \dot{y} - \dot{y}_d \\ \ddot{y} - \ddot{y}_d \\ \dot{y}^{(3)} - \dot{y}_d^{(3)} \end{pmatrix}$$

The control law (57) results in the closed-loop linear dynamics

$$\dot{z} = Az,$$

where

$$A = \begin{pmatrix} 0 & I_{2 \times 2} & 0 & 0 \\ 0 & 0 & I_{2 \times 2} & 0 \\ 0 & 0 & 0 & I_{2 \times 2} \\ -K_0 & -K_1 & -K_2 & -K_3 \end{pmatrix}$$

and so for the system to be asymptotically stable, the matrices K_i must be chosen so that A is a Hurwitz matrix. Note that with the simpler form $K_i = k_i I_{2 \times 2}$ for some $k_i > 0$, it is possible to find simple algebraic condition on the k_i 's that render A Hurwitz.

The feedback linearization form is summarized according to:

Definition	General	UAV
Outputs:	$y = h(x)$	$y = (x_1, x_2)$
Controls transformation:	$u = a(x, \xi) + B(x, \xi)v$	$u = \begin{pmatrix} \xi_1 \\ u_2 \text{ from (56)} \end{pmatrix}$
Dynamic Compensator:	$\dot{\xi} = c(x, \xi) + D(x, \xi)v$	$\dot{\xi} = \begin{pmatrix} \xi_2 \\ \ddot{u}_1 \text{ from (56)} \end{pmatrix}$

2.3 Relationship to differential flatness.

Theorem 2 ([?]). *If a control system is differentially flat then it is dynamic feedback linearizable on an open dense set, with the dynamic feedback possibly depending explicitly on time.*

In the single-input case full-state feedback linearization via static feedback and differential flatness are equivalent. In higher-dimensions, though it is more complex.

3 The general case

3.1 Single-input single-output (SISO) case

Consider the system [?]

$$\dot{x} = f(x) + g(x)u,$$

where $u \in \mathbb{R}$. Let x^* be the equilibrium, i.e. $f(x^*) = 0$. We have

$$\dot{y} = \partial h \cdot \dot{x} = \partial h \cdot [f(x) + g(x)u] \equiv L_f h(x) + L_g h(x)u,$$

where recall that $L_f h \triangleq \partial h \cdot f$. We proceed as follows:

- if $|L_g h(x)| > \delta$ for some $\delta > 0$ (i.e. bounded away from 0) then

$$u = a(x) + b(x)v = \frac{1}{L_g h(x)}(-L_f h(x) + v), \quad \dot{y} = v$$

- if $L_g h(x) = 0$, then differentiate again

$$\begin{aligned} \ddot{y} &= L_f L_f h(x) + L_g L_f h(x)u + L_f L_g h(x)u + L_g L_g h(x)u^2 \\ &= L_f^2 h(x) + L_g L_f h(x)u, \end{aligned}$$

where the last two terms dropped due to $L_g h(x) = 0$. Now if $|L_g L_f h(x)| > \delta$ for some $\delta > 0$ then

$$u = \frac{1}{L_g L_f h(x)}(-L_f^2 h + v), \quad \ddot{y} = v$$

- More generally, we keep differentiating $\dot{y}, \ddot{y}, y^{(3)}, \dots$ and denote γ to be the smallest integer for which

$$L_g L_f^i h(x) = 0, \text{ for } i = 0, \dots, \gamma - 2,$$

and $|L_g L_f^{\gamma-1} h(x)| > \delta$ (bounded away from zero). Then we have

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} (-L_f^\gamma h(x) + v), \quad y^{(\gamma)} = v,$$

i.e. the output becomes a γ -order linear system.

Definition 1. *Strict Relative Degree.* The system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

has a strict relative degree γ at x^* if

$$\begin{aligned} L_g L_f^i h(x) &= 0, \quad i = 0, \dots, \gamma - 2, \\ L_g L_f^{\gamma-1} h(x^*) &\neq 0. \end{aligned}$$

3.2 Multiple-input multiple-output system (MIMO)

We'll illustrate the MIMO case with a two-input two-output (TITO) system

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2, \\ y &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned}$$

Differentiate both until inputs start appearing, e.g. assume that at some point we have

$$\begin{aligned} y_1^{\gamma_1} &= L_f^{\gamma_1} h_1 + L_{g_1} L_f^{\gamma_1-1} h_1 u_1 + L_{g_2} L_f^{\gamma_1-1} h_1 u_2, \\ y_2^{\gamma_2} &= L_f^{\gamma_2} h_2 + L_{g_1} L_f^{\gamma_2-1} h_2 u_1 + L_{g_2} L_f^{\gamma_2-1} h_2 u_2, \end{aligned}$$

and define the matrix

$$G(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1-1} h_1 & L_{g_2} L_f^{\gamma_1-1} h_1 \\ L_{g_1} L_f^{\gamma_2-1} h_2 & L_{g_2} L_f^{\gamma_2-1} h_2 \end{bmatrix}.$$

Then we say that the system has a relative degree (γ_1, γ_2) at x^* if

$$L_{g_j} L_f^k h_i(x) = 0, \quad j = 1, 2, \quad 0 \leq k \leq \gamma_i - 2, \quad i = 1, 2$$

and the matrix G is non-singular. Then

$$u = G^{-1}(x) \left\{ \begin{bmatrix} L_f^{\gamma_1} h_1(x) \\ L_f^{\gamma_2} h_2(x) \end{bmatrix} + v \right\}, \quad \begin{bmatrix} y_1^{(\gamma_1)} \\ y_2^{(\gamma_2)} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Example 3. Consider

$$\dot{x} = \begin{bmatrix} x_2 \\ 2w(1 - \mu x_1^2)x_2 - w^2x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Let $y = x_1$, we have

$$\begin{aligned} L_g h(x) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \\ L_f h(x) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 2w(1 - \mu x_1^2)x_2 - w^2x_1 \end{bmatrix} = x_2 \\ L_g L_f h(x) &= \frac{\partial L_f h}{\partial x} \cdot g(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \end{aligned}$$

Therefore, the relative degree of the system is 2. Further since

$$L_f^2 h(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} f(x) = 2w(1 - \mu x_1^2)x_2 - w^2x_1,$$

we have

$$u = -2w(1 - \mu x_1^2) + w^2x_1 + v.$$

3.3 Normal Forms

If a SISO has a relative degree $\gamma \leq n$ at some point x^* then it can be transformed into a *normal form*, i.e. one can find a change of coordinates $x \rightarrow \Phi(x)$ such that

$$\Phi(x) = \begin{bmatrix} z \\ \eta \end{bmatrix} \triangleq \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{\gamma-1} h(x) \\ \eta_1(x) \\ \eta_2(x) \\ \vdots \\ \eta_{n-\gamma}(x) \end{bmatrix}$$

The first γ coordinates are denoted by $z \in \mathbb{R}^\gamma$, i.e. $z_1 = \Phi_1(x), \dots, z_\gamma = \Phi_\gamma(x)$. The last $n - \gamma$ coordinates $\eta \in \mathbb{R}^{n-\gamma}$ are chosen so that the following conditions hold:

1. $\Phi(x)$ is a *diffeomorphism* (defined below),
2. the dynamics of η is not directly affected by u , i.e.

$$\dot{\eta}_i(x) = \nabla \eta_i(x)^T [f(x) + g(x)u] \equiv \nabla \eta_i(x)^T f(x),$$

or in other words we must have $L_g \eta_i(x) = 0$. This last condition will enable us to express the internal dynamics as $\dot{\eta} = w(t, z, \eta)$, i.e. independently of the inputs u .

Definition 2. A *diffeomorphism* is a smooth map with smooth inverse. The implicit function theorem states that Φ is a diffeomorphism if its jacobian $\partial\Phi$ is full rank on an open set inside which the system operates.

This condition is employed to choose $\eta(x)$, by i.e. making sure that the rows of $\partial\eta(x)$ are linearly independent and that they are also linearly independent from the rows of $\partial z(x)$.

We have

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & & & \\ \vdots & I_{\gamma-1} & & \\ 0 & \cdots & 0 & \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_\gamma \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v$$

which, along with the resulting dynamics of η is written as

$$\dot{z} = A_o z + B_o v, \quad \dot{\eta} = w(t, z, \eta).$$

It can be checked that

$$\text{rank}([B_o | A_o B_o | \cdots | A_o^{\gamma-1} B_o]) = \gamma,$$

so that the system is controllable and a linear control law for the virtual input v can be chosen according to

$$v = Kz, \quad \text{where } K \text{ is such that } A_o + B_o K \text{ is Hurwitz.}$$

Finally, the virtual to physical control mapping is expressed as

$$u = \frac{1}{L_g L_f^{\gamma-1}} \left(-L_f^\gamma h(x) + v \right).$$

Although the z -dynamics is stable, the internal dynamics $\dot{\eta} = w(t, z, \eta)$ might not be.

Definition 3. If the zero dynamics

$$\dot{\eta} = w(t, 0, \eta)$$

is asymptotically stable, then the system is *minimum phase*, otherwise it is *non-minimum phase*.

Example 4. Consider the system

$$\dot{x} = \begin{bmatrix} x_3 - x_2^3 \\ -x_2 \\ x_1^2 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u, \quad \text{with output } y = x_1.$$

We have

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_3 - x_2^3 \\ \ddot{y} &= \dot{x}_3 - 3\dot{x}_2 x_2^2 \\ &= x_1^2 - x_3 + 3x_2^3 + [1 + 3x_2^2]u, \end{aligned}$$

so the relative degree is $\gamma = 2$ and hence we need only one η coordinate. We have

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= \dot{x}_1 = x_3 - x_2^3 \end{aligned}$$

We have one additional coordinate $\eta \in \mathbb{R}$. The jacobian of the coordinate transformation is

$$\partial\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3x_2^2 & 1 \\ \frac{\partial\eta}{\partial x_1} & \frac{\partial\eta}{\partial x_2} & \frac{\partial\eta}{\partial x_3} \end{bmatrix}.$$

To obtain the last function $\eta(x)$ we require that

$$\text{rank}(\partial\Phi) = 3 \quad \text{and} \quad L_g\eta = -\frac{\partial\eta}{\partial x_2} + \frac{\partial\eta}{\partial x_3} = 0$$

This is satisfied for instance for

$$\eta = x_2 + x_3.$$

The internal dynamics is

$$\begin{aligned} \dot{\eta} &= -x_2 - u + x_1^2 - x_3 + u \\ &= x_1^2 - x_2 - x_3 \\ &= z_1^2 - \eta \end{aligned}$$

The zero dynamics (i.e. at $z = 0$) then becomes $\dot{\eta} = -\eta$ so the system is minimum phase.