EN530.678 Nonlinear Control and Planning in Robotics
Lecture 7: Stabilizability and Chained Forms

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1 Introduction

The role of distributions and controllability:

- *distributions* determine possible directions of motion
- *nonlinear controllability* determines which states can be reached
- *motion planning* employs these structural properties to generate trajectories
- *trajectory tracking* processes feedback to follow these trajectories

today's slides adapted from G. Oriolo with permission

2 Stabilizability of Nonholonomic Systems

Given a nonlinear control system

$$\dot{x} = f(x, u)$$

our goal is to construct a control law

$$u = k(x)$$

which accomplishes:

- *stabilization*: an equilibrium point x_e is made asymptotically stable, or
- *tracking*: a desired feasible trajectory $x_d(t)$ is asymptotically stable
- the linear approximation of the system at x_e is

$$\dot{\delta x} = A\delta x + B\delta u \qquad \delta x = x - x_e, \delta u = u - u_e,$$

  where $A \triangleq \partial_x f(x_e, u_e)$, $B \triangleq \partial_u f(x_e, u_e)$
- if the linearized system is controllable, then the nonlinear system can be locally smoothly stabilized at x_e using a feedback law $\delta u = K\delta x$
- recall that the linear system is controllable if

$$rank([B \quad AB \quad \cdots \quad A^{n-1}B]) = n$$

- for driftless (kinematic) models $\dot{q} = G(q)u$ the linear approximation around x_e has always uncontrollable eigenvalues at zero since

$$A = 0 \quad \text{and} \quad \text{rank} B = \text{rank} G(q_e) = m \leq n$$

- Necessary conditions by *Brockett's Theorem*: If the system

$$\dot{x} = f(x, u)$$

  is locally asymptotically C^1-stabilizable at $x_e = 0$ then the image of the map

$$f : \mathbb{R}^n \times U \to \mathbb{R}^n$$

  contains some neighborhood of x_e. More formally,, $\exists \delta > 0$, s.t. $\forall \|\xi\| \leq \delta, \exists x, u$ such that $f(x, u) = \xi$.

- For the special case

$$\dot{x} = g_1(x)u_1 + ... + g_m(x)u_m$$

  with linearly independent control vectors g_i and

$$\text{rank}\{[g_1(x_e), ..., g_m(x_e)]\} = m$$

  the system is asymptotically C^1-stabilizable at x_e if and only if $m \geq n$

- Therefore, *nonholonomic mechanical systems cannot be stabilized at a point by smooth feedback*
- The alternatives are: 1) time-varying feedback; 2) non-smooth (e.g. switching) feedback

3 Steering methods for chained forms (optional material)

3.1 Overview

- the objective is to build a sequence of open-loop input commands that steer the system from q_i to q_f satisfying the nonholonomic constraints
- the degree of nonholonomy gives a good measure of the complexity of the steering algorithm
- there exist canonical model structures for which the steering problem can be solved efficiently
  - chained form
  - power form
  - Chaplygin form
- interest in the transformation of the original model equation into one of these forms
- such model structures allow also a simpler design of feedback stabilizers (necessarily, non-smooth or time-varying)
- we limit the analysis to the case of systems with two inputs, where the three above forms are equivalent (via a coordinate transformation)

3.2 Chained Forms [Murray and Sastry 1993]

- a $(2,n)$ *chained form* is a two-input driftless control system

$$\dot{z} = g_1(z)v_1 + g_2(z)v_2$$

  in the following form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1 \end{aligned}$$

- denoting the repeated Lie brackets as $\mathrm{ad}_{g_1}^k g_2$

$$\mathrm{ad}_{g_1} g_2 = [g_1, g_2], \qquad \mathrm{ad}_{g_1}^k g_2 = [g_1, \mathrm{ad}_{g_1}^{k-1} g_2]$$

  one has

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \mathrm{ad}_{g_1}^k g_2 = \begin{pmatrix} 0 \\ \vdots \\ (-1)^k \\ \vdots \\ 0 \end{pmatrix}$$

  in which $(-1)^k$ is the $(k+2)$-th entry.

- a one-chain system is completely nonholonomic (controllable) since the n vectors

$$\{g_1, g_2, ..., ad_{g_1}^i g_2, ...\}, \quad i = 1, ..., n-2$$

  are independent

- its degree of nonholonomy is $k = n - 1$
- v_1 is called the *generating input*, z_1 and z_2 are called base variables
- if v_1 is (piecewise) constant, the system in chained form behaves like a (piecewise) linear system
- chained systems are a generalization of first- and second-order controllable systems for which sinusoidal steering from z_i to z_f minimizes the integral norm of the input
- different input commands can be used, e.g.
  - sinusoidal inputs
  - piecewise constant inputs
  - polynomial inputs

3.3 Steering with polynomial inputs

- idea similar to piecewise constant input, but with improved smoothness properties w.r.t. time (remember that kinematic models are controlled at the (pseudo)velocity level)
- the controls are chosen as

$$\begin{aligned} v_1 &= \frac{z_{f1} - z_{01}}{T}, \\ v_2 &= c_0 + c_1 t + ... + c_{n-2} t^{n-2} \end{aligned}$$

  where T is desired final time and $c_0, ..., c_n$ obtained solving the linear system coming from the closed-form integration of the model

$$M(T) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix} + m(z_0, T) = \begin{pmatrix} z_{f2} \\ z_{f3} \\ \vdots \\ z_{fn} \end{pmatrix}$$

  with $M(T)$ nonsingular for $T \neq 0$.
- if $z_{f1} = z_{01}$ and intermediate point must be added

Example 1. *Unicycle: consider the following change of coordinates*

$$\begin{aligned} z_1 &= x \\ z_2 &= \tan\theta \\ z_3 &= y. \end{aligned}$$

and input variables

$$\begin{aligned} u_1 &= v_1/\cos\theta \\ u_2 &= v_2 \cos^2\theta. \end{aligned}$$

The new equivalent system becomes

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1, \end{aligned}$$

Assume that the system must move between two configurations which we epxress in terms of the new coordinates by (z_{01}, z_{02}, z_{03}) (initial) and (z_{f1}, z_{f2}, z_{f3}) (final).

To satisfy the first coordinate we set

$$v_1 = \frac{z_{f1} - z_{01}}{T}, \qquad v_2 = c_0 + c_1 t,$$

where c_0, c_1 are unknowns. After integrating $\dot{z}_2$ we have

$$z_2(t) = z_{02} + c_0 t + \frac{1}{2} c_1 t^2$$

from which after integrating $\dot{z}_3$ we get

$$z_3(t) = z_{03} + v_1 \left(z_{02}t + \frac{1}{2}c_0 t^2 + \frac{1}{6}c_1 t^3 \right)$$

Now we can solve for c_0, c_1 the relationships

$$z_2(T) = z_{f2}, \qquad z_3(T) = z_{f3}$$

which is equivalent to the relationship

$$\underbrace{\begin{bmatrix} T & \frac{1}{2}T^2 \\ v_1\frac{T^2}{2} & v_1\frac{T^3}{6} \end{bmatrix}}_{M(T)} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} + \underbrace{\begin{bmatrix} z_{02} \\ z_{03} + v_1 z_{02} T \end{bmatrix}}_{m(z_0,T)} = \begin{bmatrix} z_{f2} \\ z_{f3} \end{bmatrix}$$

and so the coefficients are found as

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = M(T)^{-1} \left(\begin{bmatrix} z_{f2} \\ z_{f3} \end{bmatrix} - m(z_0, T) \right).$$

- Sinusoidal inputs: a two-phase process
  - Phase 1: steer base variables z_1 and z_2 to their desired values z_{f1} and z_{f2}
  - Phase 2: choose

    $$\begin{aligned} v_1 &= a_0 + \sin \omega t \\ v_2 &= b_0 + \cos \omega t + ... + b_{n-2} \cos(n-2)\omega t, \end{aligned}$$

    and solve numerically for the $n+1$ unknowns in terms of the boundary conditions

- piece-wise constant controls
  - subdivide total time T into subintervals of length δ

    $$\begin{matrix} v_1(\tau) = v_{1,k} \\ v_2(\tau) = v_{2,k} \end{matrix}, \quad \tau \in [(k-1)\delta, k\delta]$$

  - it is convenient to set v_1 =constant $\Rightarrow$ then the unknowns

    $$v_{2,1}, v_{2,2}, ..., v_{2,n-1}$$

    are found by solving a triangular linear system

3.4 Chained Form Transformation

Define the distributions

$$\begin{aligned} \Delta_0 &= \text{span}\{g_1, g_2, \text{ad}_{g_1} g_2, ..., \text{ad}_{g_1}^{n-2} g_2\} \\ \Delta_1 &= \text{span}\{g_2, \text{ad}_{g_1} g_2, ..., \text{ad}_{g_1}^{n-2} g_2\} \\ \Delta_2 &= \text{span}\{g_2, \text{ad}_{g_1} g_2, ..., \text{ad}_{g_1}^{n-3} g_2\} \end{aligned}$$

If, for some open set, one has (i) $\dim\Delta_0 = n$, (ii) Δ_1, Δ_2 are involutive, (iii) there exists a function h_1 such that

$$dh_1 \cdot \Delta_1 = 0 \qquad dh_1 \cdot g_1 = 1$$

then the system can be transformed into chained form

the change of coordinates is given by

$$\begin{aligned} z_1 &= h_1 \\ z_2 &= L_{g_1}^{n-2} h_2 \\ &\vdots \\ z_{n-1} &= L_{g_1} h_2 \\ z_n &= h_2 \end{aligned}$$

with h_2 independent from h_1 and such that $dh_2 \cdot \Delta_2 = 0$ the input transformation is given by

$$v_1 = u_1 \tag{1}$$

$$v_2 = (L_{g_1}^{n-1} h_2) u_1 + (L_{g_2} L_{g_1}^{n-2} h_2) u_2 \tag{2}$$