EN530.678 Nonlinear Control and Planning in Robotics
Lecture 6: Differential Flatness

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1 Differential Flatness

Controlling a nonlinear system to achieve a desired state is accomplished through a two-step process. The first step is *trajectory generation* and the second is *trajectory tracking* using feedback. Based on computational resources available, trajectory generation can be recomputed (typically at a slow frequency) while tracking operates at the rate close to the sensor frequency. The key advantage of the two-step approach is that trajectory generation is performed to account for the system dynamics, any constraints such as input bounds, and often to minimize a desired performance metric. Thus, it is expected that even if the system deviates from the path due to uncertainties the overall performance will be close to optimal.

1.1 Basic Definition

An important class of systems for which trajectory generation is particularly easy are the so-called *differentially flat* systems [?]. A system is differentially flat if one can find a set of outputs (equal to the number of inputs) which completely determine the whole state and the inputs without the need to integrate the system. More formally, a system with state $x \in \mathbb{R}^n$ and inputs in $u \in \mathbb{R}^m$ is differentially flat if one can find outputs $y \in \mathbb{R}^m$ of the form

$$y = h(x, u, \dot{u}, ..., u^{(a)}), \tag{1}$$

such that

$$\begin{aligned} x &= \varphi(y, \dot{y}, ..., y^{(b)}), \\ u &= \alpha(y, \dot{y}, ..., y^{(c)}). \end{aligned} \tag{2}$$

The coordinates y are called *flat outputs*.

Example 1. *Fully-actuated robotic system.* Consider robotic systems with configuration $q \in \mathbb{R}^m$ and control inputs $u \in \mathbb{R}^m$ with dynamics in standard form

$$M(q)\ddot{q} + b(q, \dot{q}) = B(q)u,$$

where $B(q)$ is an invertible matrix. We have $x = (q, \dot{q})$ and consider the output function

$$h(x(t)) = q(t),$$

i.e. we have $y(t) = q(t)$. The functions φ and α then correspond to

$$\varphi(y, \dot{y}) = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad \alpha(y, \dot{y}, \ddot{y}) = B(q)^{-1}[M(q)\ddot{q} + b(q, \dot{q})], \tag{3}$$

so that we have $a = 0, b = 1, c = 2$. This means that a given trajectory $q(t)$ exactly determines the state $x(t)$ and the required controls $u(t)$.

Example 2. *Wheeled robot.* The wheeled robot is differentially flat with outputs $y = (x_1, x_2)$ by noting that

$$x_3 = atan2(\dot{x}_2, \dot{x}_1) + k\pi, \quad k = 0, 1,$$

depending on whether the vehicle is moving forward or backwards.

The control inputs are found by (recall $\frac{d}{da}\tan^{-1} a = \frac{1}{a^2+1}$)

$$\begin{aligned} u_1 &= \pm\sqrt{\dot{x}_1^2 + \dot{x}_2^2}, \\ u_2 &= \dot{x}_3 = \frac{\dot{x}_1\ddot{x}_2 - \ddot{x}_1\dot{x}_2}{\dot{x}_1^2 + \dot{x}_2^2}. \end{aligned}$$

Thus, one can generate a trajectory in position $y(t) = (x_1(t), x_2(t))$ coordinates which uniquely determines the robot orientation and required forward velocity and turn rate.

Example 3. *Planar rigid body* Consider a rigid body in the plane controlled with two body-fixed forces u_1 and u_2 and dynamics [**?**]

$$\begin{aligned} m\ddot{x}_1 &= u_1 \cos x_3 - u_2 \sin x_3 \\ m\ddot{x}_2 &= u_1 \sin x_3 + u_2 \cos x_3 - mg \\ m\ddot{x}_3 &= ru_1 \end{aligned} \tag{4}$$

The equations of motion can be combined to obtain

$$-\frac{J}{mr}\ddot{x}_3 + \ddot{x}_1 \cos x_3 + \ddot{x}_2 \sin x_3 + g \sin x_3 = 0 \tag{5}$$

It is possible to show that flat outputs y_1 and y_2 are given

$$\begin{aligned} y_1 &= x_1 - \frac{J}{mr}\sin x_3, \\ y_2 &= x_2 + \frac{J}{mr}\cos x_3. \end{aligned} \tag{6}$$

Substituting $x_1 = y_1 + \frac{J}{mr} sinx_3$, $x_2 = y_2 - \frac{J}{mr}\cos x_3$ into (5) we get

$$\ddot{y}_1 \cos x_3 + (\ddot{y}_2 + g)\sin x_3 = 0,$$

from one can determine x_3 except at the singularity $\ddot{y}_1 = \ddot{y}_2 + g = 0$. The point y is actually a body-fixed point called the center of oscillation which happens to be at distance $\frac{J}{mr}$ from the center-of-mass on the other side of the point at which control forces intersect.

1.2 Trajectory Generation

Consider the problem of generating a trajectory between two given states

$$\dot{x} = f(x, u), \qquad x(0) = x_0, x(T) = x_f.$$

The boundary conditions of a differentially flat systems are expressed as

$$\begin{aligned} x(0) &= \varphi(y(0), \dot{y}(0), ..., y^{(b)}(0)), \\ x(T) &= \varphi(y(T), \dot{y}(T), ..., y^{(b)}(T)), \end{aligned} \tag{7}$$

i.e. they are specified entirely using the flat outputs. Thus, it is possible to perform trajectory generation in the flat output space.

The general strategy is to assume a parametric form

$$y(t) = A\lambda(t),$$

where $\lambda(t) \in \mathbb{R}^N$ are basis functions and $A \in L(\mathbb{R}^m, \mathbb{R}^N)$ is matrix with constant coefficients that will be computed in order to satisfy (7). This is accomplished by computing A to satisfy the flat boundary conditions

$$y(0) = A\lambda(0), \qquad ..., \qquad y^{(b)}(0) = A\lambda^{(b)}(0), \tag{8}$$

$$y(T) = A\lambda(T), \qquad ..., \qquad y^{(b)}(T) = A\lambda^{(b)}(T), \tag{9}$$

which can be put in the compact form

$$Y = A\Lambda, \tag{10}$$

where the matrices $Y \in L(\mathbb{R}^m, \mathbb{R}^{(2(b+1)})$ and $\Lambda \in L(\mathbb{R}^N, \mathbb{R}^{2(b+1)})$ are defined by

$$Y = \left[y(0) \mid \cdots \mid y^{(b)}(0) \mid y(T) \mid \cdots \mid y^{(b)}(T) \right],$$

and

$$\Lambda = \left[\lambda(0) \mid \cdots \mid \lambda^{(b)}(0) \mid \lambda(T) \mid \cdots \mid \lambda^{(b)}(T) \right].$$

Thus, the coefficients A must lie in the affine subspace defined by (10). To satisfy the boundary conditions it is necessary that $N \geq 2(b+1)$.

In particular, for $N = 2(b+1)$ and Λ full-rank one can directly compute

$$A = Y\Lambda^{-1}.$$

Example 4. *Fully-actuated robot (continued).* The state is $x = (q, \dot{q})$ while the controls depend on $(q, \dot{q}, \ddot{q})$ and so we have $b = 1$ and $c = 2$. Assume we are interested in moving between two states x_0 and x_f in time T. We can choose basis functions according to

$$\lambda(t) = [t^3, t^2, t^1, 1]^T,$$

in which case A is a mx4 matrix of unknown constants,

$$Y = [q_0, \ \dot{q}_0, \ q_f, \ \dot{q}_f],$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 & T^3 & 3T^2 \\ 0 & 0 & T^2 & 2T \\ 0 & 1 & T & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We find the trajectory $y(t) = A\lambda(t)$ by setting $A = Y\Lambda^{-1}$. Given $y(t)$ we can find $x(t)$ and $u(t)$ as specified by (3).

Example 5. *Wheeled robot (continued).* For the wheeled robot, the state x depends on the the flat output y and its derivative $\dot{y}$ while the controls depend also on $\ddot{y}$. Therefore, we have $b = 1$ and $c = 2$. Assume we are interested in moving between two poses x_0 and x_f in time T. We can choose basis functions according to

$$\lambda(t) = [t^3, t^2, t^1, 1]^T,$$

in which case A is a 2x4 matrix of unknown constants,

$$Y = [y_0,\ \dot{y}_0,\ y_f,\ \dot{y}_f]\,,$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 & T^3 & 3T^2 \\ 0 & 0 & T^2 & 2T \\ 0 & 1 & T & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The initial and final conditions, $(y_0, \dot{y}_0)$ and $(y_f, \dot{y}_f)$ respectively are not uniquely determined by the given poses x_0 and x_f since a given pose x only specifies the attitude corresponding to the ratio $\dot{x}_2/\dot{x}_1$ rather that the velocity $\sqrt{\dot{x}_1^2 + \dot{x}_2}$. Thus, one can freely choose the starting and ending velocities $v_0 = \sqrt{\dot{x}_{10}^2 + \dot{x}_{20}}$ and $v_f = \sqrt{\dot{x}_{1f}^2 + \dot{x}_{2f}}$ in specifying $(y_0, \dot{y}_0)$ and $(y_f, \dot{y}_f)$. Such choice will affect the shape of the resulting trajectory and is useful in controlling its aggressiveness.

Example 6. *Planar rigid body (continued).* The state of the planar rigid body $x = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)$ depends on flat outputs up to order $y^{(3)}$ while the control inputs u depend on terms up to $y^{(5)}$. Therefore, we have $b = 3$ and $c = 5$. If we are interested in moving between two poses x_0 and x_f in time T we can choose basis functions according to

$$\lambda(t) = [t^7, t^6, t^5, t^4, t^3, t^2, t^1, 1]^T,$$

in which case A is a 2x8 matrix of unknown constants,

$$Y = \left[y_0,\ \dot{y}_0,\ \ddot{y}_0,\ y_0^{(3)},\ y_f,\ \dot{y}_f,\ \ddot{y}_f,\ y_f^{(3)}\right],$$

and Λ will become an invertible 8x8 matrix.

In following lectures we will introduce the concept of feedback-linearizable systems which include differentially flat systems as a special case. The trajectories generated for flat system can be tracked using the feedback linearization techniques that we will study next. The flat outputs naturally become the outputs leading to feedback linearization.