EN530.678 Nonlinear Control and Planning in Robotics
Lecture 5: Distributions and Controllability
February 24, 2021

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1 Distributions

The role of distributions and controllability:

- *distributions* determine possible directions of motion
- *nonlinear controllability* determines which states can be reached
- *motion planning* employs these structural properties to generate trajectories
- *trajectory tracking* processes feedback to follow these trajectories
- Let $g_1(x), ..., g_m(x)$ be linearly independent vector fields on M.
- A *distribution* Δ assigns a subspace of the tangent space to each point defined by

$$\Delta = \text{span}\{g_1, ..., g_m\}.$$

- A distribution Δ is *involutive* if it is closed under the Lie bracket, i.e. if

$$\forall f(x), g(x) \in \Delta(x), \quad [f(x), g(x)] \in \Delta(x)$$

- A distribution Δ is *regular* if the dimension of Δ_x does not vary with x.
- A distribution Δ of constant dimension k is *integrable* if for every $x \in \mathbb{R}^n$ there are smooth functions $h_i : \mathbb{R}^n \to \mathbb{R}$ such that $\frac{\partial h_i}{\partial x}$ are linearly independent at x and for every $f \in \Delta$

$$L_f h_i = \frac{\partial h_i}{\partial x} f(x) = 0, i = 1, ..., n-k.$$

- The hypersufraces defined as the level sets

$$\{q : h_1(x) = c_1, ..., h_{n-k}(x) = c_{n-k}\},$$

are called *integral manifolds* for the distribution.

- **Frobenius Theorem:** *A regular distribution is integrable if and only if is involutive.*
- If the distribution Δ is involutive then its integral manifolds (level sets of functions h_i) are *leaves* of a *foliation* of $\mathbb{R}^n$

Examples

- The nonholonomic integrator

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix} u_2$$

- Trapped on a sphere

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} u_2$$

2 Nonlinear Controllability

2.1 Reachable Sets

- Consider the nonlinear control system (NCS)

$$\Sigma : \qquad \dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x \in \mathbb{R}^n, u \in U \subset \mathbb{R}^m$$

- A system is *controllable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a time T and $u : [0, T] \to U$ such that Σ satisfies $x(0) = x_0$ and $x(T) = x_f$.

- A system is *small-time locally controllable* (STLC) at x_0 if it can reach nearby points in arbitrary small times and stay near x_0.

- The *reachable set* $\mathcal{R}^V(x_0, T)$ is the set of states $x(T)$ for which there is a control $u : [0, T] \to U$ that steers the system from $x(0)$ to $x(T)$ without leaving an open set V around x_0.

- The set of states reachable *up to* time T is defined by

$$\mathcal{R}^V(x_0, \leq T) = \bigcup_{0 < \tau \leq T} \mathcal{R}^V(x_0, \tau)$$

2.2 Controllability Conditions

- NCS is *locally accessible* (LA) from x_0 if $\forall V$, a neighborhood of x_0 and $\forall T > 0$

$$\Omega \subset \mathcal{R}^V(x_0, \leq T), \text{ for some open set } \Omega$$

- NCS is STLC if every neighborhood V of x_0 and every $T > 0$ if $\mathcal{R}^V(x_0, T)$ contains a neighborhood of x_0.

- STLC $\Rightarrow$ controllability $\Rightarrow$ LA (not vice versa)

- Checking LA is performed through an *algebraic test*:
  - Let $\bar{\Omega}$ be the involutive closure of the distribution of $\{g_0, g_1, ..., g_m\}$
  - **Theorem (Chow):**. NCS is LA from x_0 if and only if

$$\dim\overline{\Delta}(x_0) = n : \quad \textit{accessibility rank condition}$$

  - Algorithmic Test:

$$\bar{\Delta} = \text{span}\left\{ v \in \bigcup_{k\geq 0} \Delta^k \right\} \text{ with } \left\{ \begin{array}{l} \Delta^0 = \text{span}\{g_0, g_1, ..., g_m\} \\ \Delta^k = \Delta^{k-1} + \text{span}\{[g_j, v], j = 0, ..., m : v \in \Delta^{k-1}\} \end{array} \right.$$

- only sufficient conditions exists for STLC , e.g., [Sussmann 1987]
- however, for driftless control systems:

$$\text{LA} \Leftrightarrow \text{controllability} \Leftrightarrow \text{STLC}$$

- this equivalence holds also whenever

$$g_0(x) \in \text{span}\{g_1(x), ..., g_m(x)\}, \qquad \forall x \in X$$

  (“trivial” drift)
- if the driftless control system

$$\dot{q} = \sum_{i=1}^{m} g_i(q) v_i,$$

  with state q and inputs v is controllable, then its *dynamic extension*

$$\dot{q} = \sum_{i=1}^{m} g_i(x) v_i,$$
$$\dot{v}_i = u_i, \quad i = 1, ..., m,$$

  with state $x = (q, v)$ and controls u is also controllable (and vice versa).

Examples

- The unicycle

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow g_3 = [g_1, g_2] = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

  $\dim\bar{\Delta} = 3$ for all q

- The car-like robot (rear-drive)

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ -1/\ell\cos^2\phi \\ 0 \end{pmatrix}, \ g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell\cos^2\phi \\ \cos\theta/\ell\cos^2\phi \\ 0 \\ 0 \end{pmatrix}$$

$\dim\bar{\Delta} = 4$ away from singularity at $\phi = \pm\pi/2$ of g_1

- more generally, the *filtration* of a distribution Δ is defind by

$$\Delta_1 = \Delta, \qquad \Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}], i \geq 2$$

where

$$[\Delta_1, \Delta_{i-1}] = \mathrm{span}\{[g, h] : g \in \Delta_1, h \in \Delta_{i-1}\}$$

- after enough "bracketing" (e.g. k times) the rank of Δ_i for $i \geq k$ stops increasing, no more new directions of motion appear. The smallest such k is called *degree of nonholonomy* of the distribtion, i.e. such that

$$\dim\Delta_{k+1} = \dim\Delta_k.$$

- Classification in terms of k
  - completely nonholonomic: $\dim(\Delta_k) = n$
  - partially nonholonomic: $m < \dim(\Delta_k) < n$
  - holonomic: $\dim(\Delta_k) = m = n - k$

- Examples: unicycle ($k = 2$), car-like robot ($k = 3$)

2.3 Good and bad brackets

For the general system with non-zero drift g_0 term we will use the concept of good and bad brackets.

A *bad bracket* is a Lie bracket generated using an odd number of g_0 vectors and even number of g_i (for each $i = 1, \ldots, m$) vectors. A *good bracket* is one that is not bad.

Theorem 1. *A control system with $x \in \mathbb{R}^n$ and controls $u \in U \subset \mathbb{R}^m$*

$$\dot{x} = g_o(x) + \sum_{i=1}^{m} g_i(x) u_i$$

is STLC at x^* if

1. $g_0(x^*) = 0$

2. *U is open and its convex hull contains* 0

3. *LARC is satisfied using brackets of degree k*

4. *any bad bracket of degree $j \leq k$ can be expressed as linear combination of good brackets of degree $< j$*

Example 1. *from Principles of Robot Motion Consider the planar rigid body with state $x \in \mathbb{R}^6$ defining its position, orientation, and velocities, controls $u \in \mathbb{R}^2$ defining the forward force and lateral force (at distance d from the center-of-mass) with vector fields*

$$g_0(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ 0 \\ 0 \end{pmatrix}, g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}, g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin x_3 \\ \cos x_3 \\ -d \end{pmatrix},$$

We can define

$$g_3 = [g_0, g_1], g_3 = [g_0, g_2], g_5 = [g_1, g_4], g_6 = [g_0, g_5],$$

and note that

$$det([g_1, g_2, g_3, g_4, g_5, g_6]) = d^4 \Rightarrow \text{ LARC of degree 4}$$

The bad brackets of degree ≤ 4 are

$$[g_1, [g_0, g_1]] = 0, \quad [g_2, [g_0, g_2]] = (0, 0, 0, 2d\cos x_3, 2d\sin x_3, 0) \triangleq 2dg_1,$$

and since both are spanned by good brackets of lower order then the system is STLC. Note that since the first bad bracket is zero then it becomes irrelevant. We didn't have to consider brackets of oder 4 since by definition they are good (i.e. since all *control vector fields must appear even number of times and the drift appear odd number of times then bad brackets must have odd order).*