EN530.678 Nonlinear Control and Planning in Robotics
Lecture 4: Manifolds and Vector Fields

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1 Manifolds

In many practical applications the inherent nature of the configuration space is different than $\mathbb{R}^n$. For instance:

- a manipulator with rotational joints lives on a torus $T^m = S^1 \times ... \times S^1$
- rigid body configuration space of frames SE(3)
- end-effector might be constrained to a sphere S^2
- Physical constraints
  - *position constraints*: contact (e.g. end-effector constrained to a surface), mechanical joints and other kinematic coupling relationships.
  - *rolling and sliding* (car cannot move sideways, knife-edge can slide forward). These are velocity constraints that result in *integral manifolds*, i.e. from integrating flow along the subspace of allowable velocities.
  - *non-smooth constraints, hybrid systems*: end-effector move freely, then lands on a rigid surface and slides; legged robot stance on a surface, leaving/landing on the surface.
- *sensing constraints*, e.g. maintain line-of-sight to a point
- in computer science: compression of high-dimensional data, i.e. to identify and model *lower-dimensional structure*. But it is used very loosely.
- mathematical physics, general relativity, etc...

A *manifold* is a set M that locally "looks like" linear space, e.g. $\mathbb{R}^n$. A *chart* on M is a subset U of M together with a bijective map $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. We usually denote $\varphi(m)$ by $(x^1, \dots, x^n)$ and call the x^i the coordinates of the point $m \in U \subset M$. Two charts U, φ and U', φ' such that $U \cap U' \neq \emptyset$, are called *compatible* if $\varphi(U \cap U')$ and $\varphi(U' \cap U)$ are open subsets of $\mathbb{R}^n$ and the maps

$$\varphi' \circ \varphi^{-1}|_{\varphi(U\cap U')} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

and

$$\varphi \circ (\varphi')^{-1}|_{\varphi'(U\cap U')} : \varphi'(U \cap U') \to \varphi(U \cap U')$$

are C^∞ (smooth). Here $\varphi \circ (\varphi')^{-1}|_{\varphi'(U\cap U')}$ denotes the restriction of the map $\varphi \circ (\varphi')^{-1}$ to the set $\varphi'(U \cap U')$.

We call M a *differentiable n-manifold* when:

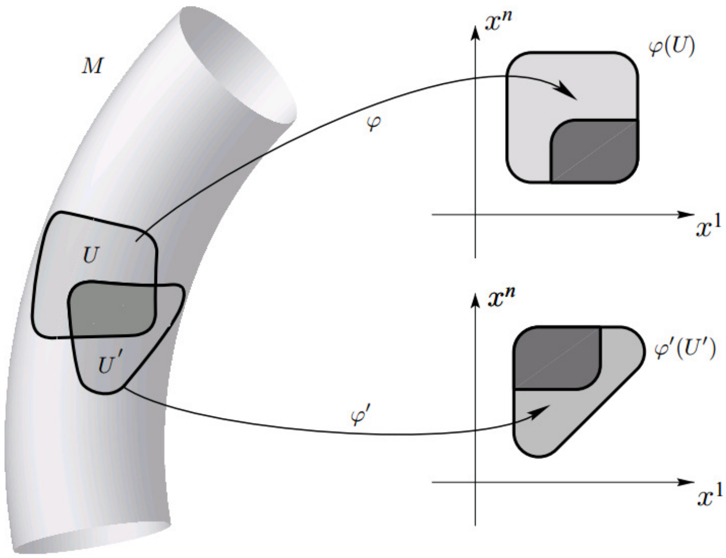

Figure 1: A manifold must be covered my overlapping charts and smooth transitions between them.

1. The set M is covered by a collection of charts, that is, every point is represented in at least one chart
2. M has an *atlas*; that is, M can be written as a union of compatible charts

In practice, the requirement that $\varphi' \circ \varphi^{-1}|_{\varphi(U \cap U')}$ and $\varphi \circ (\varphi')^{-1}|_{\varphi'(U \cap U')}$ both are both smooth and map to open sets can be translated into checking that the Jacobian of these maps is full rank.

For example, consider $\mathbb{R}^3$ as a manifold. First, we pick standard cartesian coordinates (i.e. the chart is the identity), but then add other charts such as spherical coordinates – then the collection becomes a differentiable structure (i.e. one can pass from one chart to the other smoothly). This will be understood to have been done when we say we have a manifold.

Example 1. *The circle* S^1 is the set of all points (x, y) such that $x^2 + y^2 = 1$. We can show that it is a manifold with charts e.g. $\varphi : S^1 \backslash (-1, 0) \to (-\pi, \pi)$, and $\varphi' : S^1 \backslash (0, -1) \to (-\pi/2, \frac{3}{2}\pi)$:

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \operatorname{atan2}(y, x) \triangleq \theta, \quad \varphi^{-1}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

where $\operatorname{atan2}(y, x) \in (-\pi, \pi]$ and is undefined for $x = y = 0$. A second chart is defined by "rotating" counter-clock-wise U by $\pi/2$ radians to obtain U' (the resulting coordinate θ' in U' is then obtained by subtracting $\pi/2$ from the coordinate θ on U) :

$$\varphi'\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \operatorname{atan2}(y, x) - \pi/2 \triangleq \theta', \quad \varphi'^{-1}(\theta') = \begin{bmatrix} \cos(\theta' + \pi/2) \\ \sin(\theta' + \pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\theta' \\ \cos\theta' \end{bmatrix}.$$

Note that we then have $\varphi' \circ \varphi^{-1}(\theta) = \theta - \pi/2$ and $\varphi \circ (\varphi')^{-1}(\theta') = \theta' + \pi/2$ which are smooth and compatibale (i.e. differentiable with full-rank Jacobians on the overlapping subset of the manifold) maps. The Jacobian of both maps is just 1.

Example 2. *S^2 with spherical coordinates*

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \arccos(z) \\ \mathrm{atan2}(y, x) \end{bmatrix}, \quad \varphi^{-1}\left(\begin{bmatrix} \theta \\ \varphi \end{bmatrix}\right) = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$$

Note we have $U = M \backslash \{(-\sqrt{1-z^2}, 0, z) \mid z \in [-1, 1]\}$ which represents the whole sphere with an arc connecting the two poles removed (the arc passes through $x = -1, y = 0, z = 0$). This arc needs to be removed for the same reason why we needed to remove the point $(-1, 0)$ in the S^1 example and the reason why the poles themselves cannot be included is because the chart has a singularity at $x, y = 0$. A second chart can be constructed by "rotating" U e.g. by $\pi/2$ radiauns around the y-axis and then rotating it by e.g. π radias around the z-axis. As a result the two removed arcs will not overlap and thus we will cover the full space.

We can also show that S^2 is a manifold using *stereo-graphic projection* charts.

Example 3. *S^n with stereo-graphic coordinates*

We first consider the regular sphere S^2 embedded in $\mathbb{R}^3$. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the north and south poles of the sphere. We can define $U = \mathbb{R}^3 \backslash N$ and $U' = \mathbb{R}^3 \backslash S$ and mappings

$$\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \qquad \varphi'(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

corresponding to the projection a point $m = (x, y, z)$ from the north (resp. south) poles onto the plane tangent to the south (resp. north) poles.

The inverse of these mappings is given by

$$\varphi^{-1}(u_1, u_2) = \left(\frac{2u_1}{1+\|u\|^2}, \frac{2u_2}{1+\|u\|^2}, \frac{-1+\|u\|^2}{1+\|u\|^2}\right), \varphi'^{-1}(u_1', u_2') = \left(\frac{2u_1'}{1+\|u'\|^2}, \frac{2u_2'}{1+\|u'\|^2}, -\frac{-1+\|u'\|^2}{1+\|u'\|^2}\right)$$

, while their composition is

$$\varphi' \circ \varphi^{-1}(u) = \left(\frac{u_1}{\|u\|^2}, \frac{u_2}{\|u\|^2}\right)$$

which we can show is differentiable.

The construction above extends to the n-sphere S^n by computing the last coordinate similar to z for S^2, i.e. for a point $m \in S^2$ we have

$$\varphi(m) = \left(\frac{m_1}{1-m_n}, \dots, \frac{m_{n-1}}{1-m_n}\right)$$

and so on, analogously to the S^2 case described above.

Definition 1. *Tangent Vectors. Two curves $t \to c_1(t)$ and $t \to c_2(t)$ in an n-manifold M are called equivalent at the point m if*

$$c_1(0) = c_2(0) = m,$$

and

$$\frac{d}{dt}(\varphi \circ c_1)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)\Big|_{t=0}$$

in some chart φ.

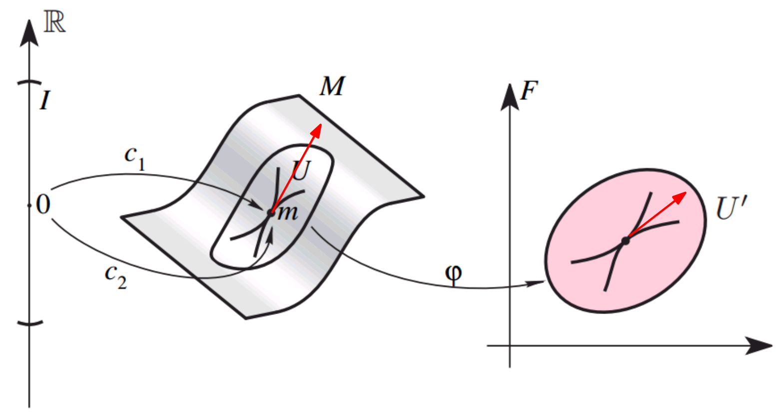

Figure 2: Consider two curves c_1 and c_2 passing through a point m on the manifold M with the same velocity vector at that point. Such velocity vectors are called tangent vectors.

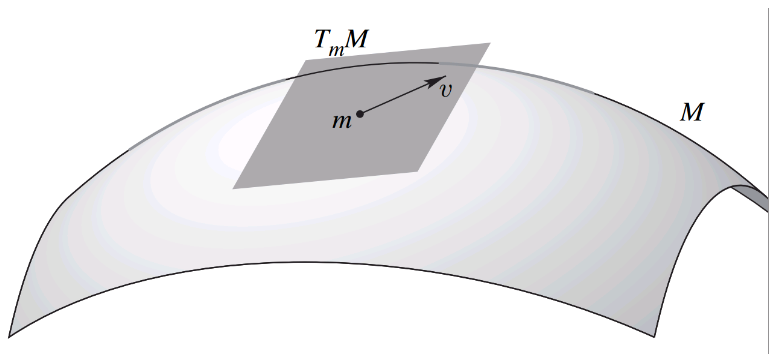

Figure 3: All tangent vectors at a point m form a vector space (e.g. they could span $\mathbb{R}^n$), called the tangent space at m.

A *tangent vector* v to a manifold M at point m is an equivalent class of curves at m. The set of tangent vectors to M at m is a vector space. We denoted it by T_mM = *tangent space* to M at $m \in M$. We think of $v \in T_mM$ as tangent to a curve in M.

The *components* of a tangent v are the numbers $v^1, \ldots, v^n$ defined by taking derivatives of the components of the curve $\varphi \circ c$:

$$v^i = \frac{d}{dt}(\varphi \circ c)^i\Big|_{t=0}$$

The components are independent of the representative curve chosen, but they do depend on the chart chosen. (Think of *components* as the coordinates of the velocity).

Definition 2. *The tangent bundle* of M denoted by TM is the disjoint union of the tangent spaces to M at the points $m \in M$, i.e.

$$TM = \cup_{m \in M} T_mM$$

Points in TM are vectors v tangent at some $m \in M$. If M is an n-manifold then TM is a $2n$-manifold. The *natural projection* is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $m \in M$ at which the vector v is attached. The inverse image $\tau_M^{-1}(m)$ of $m \in M$ is the tangent space $T_m M$ – the *fiber* of TM over the point $m \in M$.

1.0.1 Vector fields.

Definition 3. *A* vector field *X on M is a map $X : M \to TM$ that assignes a vector $X(m)$ at the point $m \in M$, i.e. $\tau_M \circ X =$ identity. The vector space of vector fields is denoted $\mathfrak{X}(M)$.*

An *integral curve* of X with initial condition m_0 at $t = 0$ is a map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = m_0$ and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$, i.e. a *solution curve* of this ODE.

The *flow* of X: a collection of maps $\Phi_t : M \to M$ such that $t \to \Phi_t(m)$ is the integral curve of X with initial condition m.

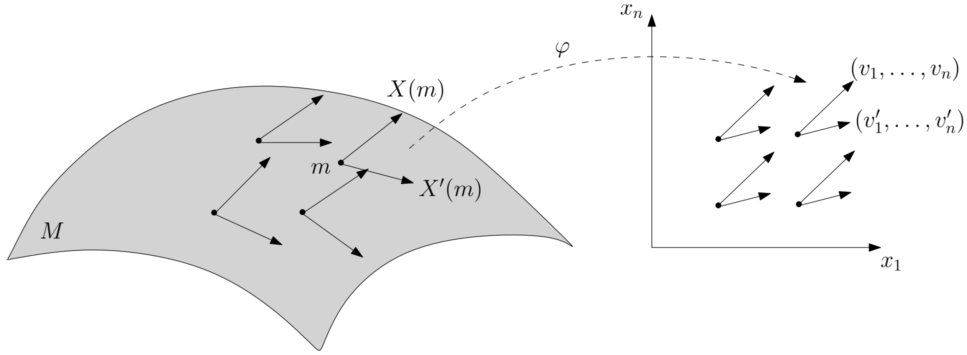

Figure 4: Two vector fields X and X' represented by local coordinates $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$.

The deriviative of $f : M \to \mathbb{R}$ at $m \in M$ gives a map $T_m f : T_m M \to T_{f(m)}\mathbb{R} \simeq \mathbb{R}$. It is actually a linear map $df(m) : T_m M \to \mathbb{R}$. Thus $df(m) \in T_m^* M$, the dual of $T_m M$ (the dual is the space of linear functions). If we replace each vector space $T_m M$ with its dual $T_m^* M$ we obtain a $2n$-manifold called the cotangent bundle and denoted by T^*M. We call *df* the *differential* of f. For every $v \in T_m M$ we call $df(m) \cdot v$ the *directional derivative* of f in the direction of v.

In a coordinate chart or in a vector space, this notion conincides with the usual notion of a directional derviative learned in vector calculus. Using a chart φ the directional derivative is

$$df(m) \cdot v = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} v^i$$

Note that with this definition we can regard vectors v as *differential operators*, i.e. which differentiate functions. In particular we can identify a basis of $T_m M$ using the operators $\frac{\partial}{\partial x^i}$ and we write

$$\{e_1, \ldots, e_n\} = \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$$

for this basis, so that $v = v^i \frac{\partial}{\partial x^i}$. In other words, think of $\frac{\partial}{\partial x^1}$ as a unit column vector $(1, 0, \ldots, 0)$ along which we can differentiate.

Tangent vectors in one chart transform to tangent vectors in another chart through the Jacobian of the map between the two charts.

Example 4. Vector field on a sphere using spherical coordinates (θ, ϕ). *An example of a vector field in a basis* $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi})$ *is e.g.* $X = \theta^2 \frac{\partial}{\partial\theta} - \theta\phi\frac{\partial}{\partial\phi}$. *It can be visualized using the standard basis in* $\mathbb{R}^3$ *defined using the Jacobian*

$$D\varphi^{-1} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix}$$

the columns of which span a tangent space at each $q \in S^2$. *So the vector field* X *expressed in cartesian coordinates will look like the vector*

$$\begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \theta^2 \\ -\theta\phi \end{bmatrix}.$$

Note: at $\theta = (0, \pi)$ *the tangent space is undefined* $\Rightarrow$ *need two charts.*

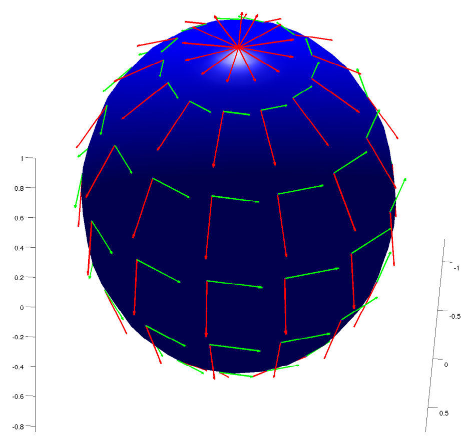

Figure 5: Tangent vectors on the sphere corresponding to basis vectors $(1, 0)$ and $(0, 1)$ in spherical coordinates at each piont $(x^1, x^2) = (\theta, \phi)$. These tangent vectors were computed using the jacobian of the inverse of the coordinate map φ. Note that one of the basis vectors shrinks to zero at the poles, suggesting that one chart is not enough to cover the sphere.

There is a one to one correspondence between vector fields X on M and the differential operators

$$X[f](m) = df(m) \cdot X(m)$$

The *dual basis* to $\frac{\partial}{\partial x^i}$ is denoted by dx^i (think row unit vector so that $dx^j \frac{\partial}{\partial x^i} = 1$ only when $i = j$ and 0 otherwise). Thus, relative to a choice of local coordiantes we get the basis formula

$$df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

We also have

$$X[f] = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}$$

which is why we write

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

The *Lie derivative* $L_X f$ is another commonly used notation, i.e. at a point $m \in M$

$$L_X f(m) \equiv X_m f \equiv X[f](m).$$

Example 5. *In local spherical coordinates in* S^2 *from Example 4 the Lie derivative of a given function* $\alpha : S^2 \to \mathbb{R}$ *is expressed as*

$$X\alpha \equiv L_X \alpha = \theta^2 \frac{\partial \alpha}{\partial \theta} - \theta\phi \frac{\partial \alpha}{\partial \phi}.$$

Example 6. Gradient vector field *For any given function* $\alpha : M \to \mathbb{R}$ *it is possible to construct a vector field using its gradient, i.e.*

$$X = \nabla \alpha = \left(\frac{\partial \alpha}{\partial x_1}, ..., \frac{\partial \alpha}{\partial x_1} \right).$$

Example 7. Dynamics and Lyapunov function *Example: we already saw that for* $\dot{x} = f(x)$, $\dot{V} = \frac{\partial V}{\partial x} f(x) \equiv L_f V$*, the derivative of* V *in the direction of the dynamics.*

1.0.2 Lie bracket

Q: Given two vector fields $g_1(x)$ and $g_2(x)$ do their flows commute $\Phi_t^{g_2} \circ \Phi_t^{g_1} = \Phi_t^{g_1} \circ \Phi_t^{g_2}$?

A: In general, no. They bend and twist, unless they are constant vectors.

This is quantified by

$$\Phi_t^{-g_2} \circ \Phi_t^{-g_1} \circ \Phi_t^{g_2} \circ \Phi_t^{g_1}(x_0) = x_0 + t^2[g_1, g_2] + O(t^3)$$

The *Lie bracket* of two vector fields g_1 and g_2 denoted by $[g_1, g_2]$ is a new vector field defined by

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2. \tag{1}$$

or as applied to a function α by

$$[g_1, g_2]\alpha = g_1(g_2\alpha) - g_2(g_1\alpha),$$

The Lie bracket tells us what direction of motion occurs after sequencing motions along the two vector fields g_1 and g_2. To prove (1) we can perform Taylor series expansion of the flows, as follows:

$$\begin{aligned}\Phi_t^{g_1}(x_0) &= x_0 + t\dot{\Phi}_t^{g_1}(x_0) + \frac{1}{2}t^2\ddot{\Phi}_t^{g_1}(x_0)\\ &= x_0 + tg_1(x_0) + \frac{1}{2}t^2\frac{\partial g_1}{\partial x}g_1(x_0) + O(t^3),\end{aligned}$$

where we used the fact that $\dot{\Phi}_t^{g_1} = g_1$ and $\dot{g}_1(x) = \frac{\partial g_1}{\partial x}\dot{x}$, where the velocity is $\dot{x} = g_1$. Next let's denote the resulting state after the first flow by $x_1 = \Phi_t^{g_1}(x_0)$. Now, combining this flow with the flow along g_2 we have:

$$\begin{aligned}
\Phi_t^{g_2} \circ \Phi_t^{g_1}(x_0) &= x_0 + tg_1(x_0) + \frac{1}{2}t^2\frac{\partial g_1}{\partial x}g_1(x_0) + t\dot{\Phi}_t^{g_2}(x_1) + \frac{1}{2}t^2\ddot{\Phi}_t^{g_2}(x_1) + O(t^3) \\
&= x_0 + tg_1(x_0) + \frac{1}{2}t^2\frac{\partial g_1}{\partial x}g_1(x_0) + tg_2(x_0) + t^2\frac{\partial g_2}{\partial x}g_1(x_0) + \frac{1}{2}t^2\frac{\partial g_2}{\partial x}g_2(x_0) + O(t^3) \\
&= x_0 + t\left(g_1(x_0) + g_2(x_0)\right) + \frac{t^2}{2}\left(\frac{\partial g_1}{\partial x}g_1(x_0) + \frac{\partial g_2}{\partial x}g_2(x_0) + 2\frac{\partial g_2}{\partial x}g_1(x_0)\right) + O(t^3)
\end{aligned}$$

Now, if we do the same procedure but switch the order of the flows, i.e. using $\Phi_t^{g_1} \circ \Phi_t^{g_2}(x_0)$, and take the difference of the two results we would obtain exactly the expression (1) for that difference.

Since $g_1\alpha$ denotes the directional derivative of a function α in the direction generated by g_1, then the Lie bracket $[g_1, g_2]$ can be regarded as the directional derivative of a vector field g_2 in the direction generated by g_1.

The bracket has a special role – together with the linear space of vector fields at a point it forms an algebra. More specifically, a vector space V with a bilinear operator $[\cdot,\cdot] : V \times V \to V$ satisfying the following properties

1. Skew-symmetry: $[v\, w] = -[w, v]$ for all $v, w \in V$
2. Jacobi identity:
$$[[v, w], z] + [[z, v], w] + [[w, z], v] = 0,$$
   for all $v, w, z \in V$ is a *Lie algebra.*

Example 8. *The unicycle*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 \tag{2}$$

Example 9. *Euclidean space with cross-product:* $(\mathbb{R}^3, [v, w] = v \times w)$*, is a Lie algebra (e.g. applications in* $SO(3)$*)*

Other Examples:
Vector space $(V, [\cdot,\cdot] = 0)$, abelian Lie algebra
Matrix group $(GL_n, [A, B] = AB - BA)$, is a Lie algebra