EN530.678 Nonlinear Control and Planning in Robotics
Lecture 3: Stability

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0.1 Model prerequisites

Consider $\dot{x} = f(t, x)$. We will make the following basic assumptions ensuring that this model can be used for evolving the state x: $f(t, x)$ is piecewise continuous in t and locally Lipschitz, i.e.:

- $f(t, x)$ is *piecewise continuous* if f is continuous on any subinterval of t except at, possibly, finite number of points where it might have finite-jump discontinuitiies
- $f(t, x)$ is *locally Lipschitz* on a domain $D \subset \mathbb{R}^n$ if for all $x_0 \in D$ there is a neighborhood $B_r(x_0) \triangleq \{x \in \mathbb{R}^n | \|x - x_0\| \leq r\}$ around x_0 which satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \qquad L > 0,$$

for all $x, y \in B_r(x_0)$.

A point $x^* \in \mathbb{R}^n$ is an *equilibrium point* of the system if $f(t, x^*) = 0$. We would be interested in controlling the system to such points. When the goal is to regulate/stabilize the system to a given x^* we could awlays transform the problem to stabilizing to the origin, by shifting the coordinate system by x^*.

0.2 Stability

Definition 1. Stability in the sense of Lyapunov. *An equilibirum point $x_0 = 0$ is* stable *at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that*

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t > t_0. \tag{1}$$

Stability is defined at time instant t_0. *Uniform stability* further guarantees that stability holds for all t_0.

Definition 2. Asymptotic Stability. *An equilibirum point $x_0 = 0$ is* asymptotically stable *at $t = t_0$ if it is stable and* locally attractive *, i.e. there exists a $\delta(t_0)$ such that*

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0. \tag{2}$$

When stability holds for any $t > t_0$ it is called *uniform stability.* When it holds for all initial $x \in \mathbb{R}^n$ it is called *global*, otherwise it is *local.*

Asymptotic stability does not provide information about how quickly the system approaches equilibrium. This notion is quantified by exponential stability.

Definition 3. Exponential Stability. *An equilibirum point* $x_0 = 0$ *is* exponentially stable *if there exist constants* $m, \alpha, \epsilon > 0$ *such that*

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)} \|x(t_0)\|,$$

for all $\|x(t_0)\| \leq \epsilon$ *and* $t > t_0$. *The largest constant* α *is called the* rate of convergence.

0.3 Autonomous Systems

We first consider *autonomous systems*, i.e. for which the dynamics does not depend on time and can be generally expressed as

$$\dot{x} = f(x).$$

There are two general methods for stability analysis: direct and indirect. The direct method works directly with the nonlinear dynamics by seeking an energy-like function called Lyapunov function. The Lyapunov function has a minimum at the equilibrium and never increases along trajectories which corresponds to a stable motion (otherwise if the system is departing from equilibirum this energy would grow). The argument generalizes the notion of stability even for non-energetic systems such as a financial portfolio. The indirect method is based on linearization around the equilibirum and can be used to determine stability only in the vicinity of the equilibrium.

0.3.1 Lyapunov Direct Method

The method was originally proposed by Lyapunov around 1890 for studying local stability and later extended to the global setting. Let $V(x)$ be continuously differentiable function defined over $D \subset \mathbb{R}^n$, $0 \in D$.

Theorem 1. *Lyapunov's Theorem.* If there is a $V(x)$ such that a

$$V(0) = 0 \quad \text{and } V(x) > 0, \forall x \in D/\{0\}$$
$$\dot{V}(x) \leq 0, \quad \forall x \in D,$$

then the origin is *stable*. If $\dot{V}(x) < 0, \quad \forall x \in D/\{0\}$ then it is *asymptotically* stable. Furthermore, if $V(x) > 0$ for all $x \neq 0$,

$$\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty,$$

(i.e. V is *radially unbounded*) and $\dot{V}(x) < 0, \forall x \neq 0$ then the origin is globally asymptotically stable.

We have the following definitions for a function V:

$V(0) = 0, V(x) \geq 0, \forall x \neq 0$	positive semidefinite (p.s.d.)
$V(0) = 0, V(x) > 0, \forall x \neq 0$	positive definite (p.d.)
$V(0) = 0, V(x) \leq 0, \forall x \neq 0$	negative semidefinite (n.s.d.)
$V(0) = 0, V(x) < 0, \forall x \neq 0$	negative definite (n.d.)
$\|x\| \to \infty \Rightarrow V(x) \to \infty$	radially unbounded

For instance, let $x \in \mathbb{R}^2$ so that $x = (x_1, x_2)$. Then the function $V(x) = x^T x$ is p.d. but $V(x) = x_1^2$ is p.s.d. Similarly, $V(x) = -x^T x$ is n.d. but $V(x) = -x_1^2$ is n.s.d.

The theorem can be equivalently stated as follows [**?**]: *the origin is stable if there is a continuously differentiable positive definite function* $V(x)$ *so that* $\dot{V}(x)$ *is negative semidenite, and it is* *asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.*

The function V satisfying the conditions for stability is called a *Lyapunov function.* The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface*, or *level surface.*

Geometric Interpretation: Consider a level set $V(x) = c$. At point x we have:

$$\dot{V} = \nabla V^T \dot{x},$$

If $\dot{x}$ and ∇V point in the same direction then $\dot{V} > 0$. This means that if a system is stable, then trajectories should cross level sets only inwards. To find the region of stability, the region can be expanded until $\dot{V} \geq 0$ is detected, or equivalently if V fails to strictly increase.

Theorem 2. *Local instability.* Let $x = 0$ be an equilibrium point. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $\dot{V}(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Let $V(x) > 0$ in a ball B_r around 0. Then, $x = 0$ is unstable.

0.3.2 Lyapunov's indirect method

Assume that the system is linearized around the equilibrium $x_0 = 0$,

$$\dot{x} = f(x) = Ax + h(x),$$

where $A = \partial_x f|_{x=0}$ and $h(x)$ defines the nonlinear terms.

Theorem 3. Stability by linearization. *If the origin 0 is an asymptotically stable equilibirum of*

$$\dot{z} = Az,$$

(equivalently if A is Hurwitz i.e. $Re\lambda_i(A) < 0$ for all i) and h is well-behaved, i.e.

$$\lim_{\|x\| \to 0} \frac{h(x)}{\|x\|^{1+p}} = 0, \text{ for some } p \geq 0,$$

then it is a locally asymptotically stable equilibirum point of $\dot{x} = f(x)$. Furthermore, if $Re\lambda_i(A) > 0$ for any i, then the system is unstable.

We cannot conclude anything for the case when $Re\lambda_i(A) \leq 0$ for all i, or $Re\lambda_i(A) = 0$ for some i.

For 2-D systems it is instructive to study the behavior around critical points

- Critical points (show phase portraits)

Critical Point	Behavior	Eigenvalues
Stable node	stable	all real and negative
Untable node	unstable	all real and positive
Saddle point	unstable saddle	all real, positive and negative
Stable focus	damped oscillations	both complex, negative real parts
Unstable focus	undampled oscillations	both complex, positive real parts
Centre	concentric ellipses	both complex, zero real parts

The procedure is:

1. Find all critical points of $\dot{x} = f(x)$, denoted by x^*
2. Linearize at each critical point x^*: $\dot{x} = Ax + h(x)$, where $A \triangleq \partial f|_{x=x^*}$.
3. The behavior of the nonlinear system near x^* is determined by A if:
   - no eigenvalues of A have zero real parts
   - h is well-behaved

Example 1. Consider the system $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_1^2 - x_2$. The critical values are $(0,0)$ and $(-1,0)$. The linearization is

$$Df(x) = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1 & -1 \end{bmatrix}$$

which is evaluted at each ciritical point according to

$$Df|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad Df|_{x=(-1,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda_{1,2} = -.5 \pm i\sqrt{3}/2$ at $(0,0)$ and $\lambda_1 \approx -1.618, \lambda_2 \approx 0.618$ at $(-1,0)$. Thus, the first equilibirum is a stable focus, while the second is saddle point. We piece together these local behaviors to infer the behavior more globally. In particular, the structure between the two equilibria called a *separatrix*.

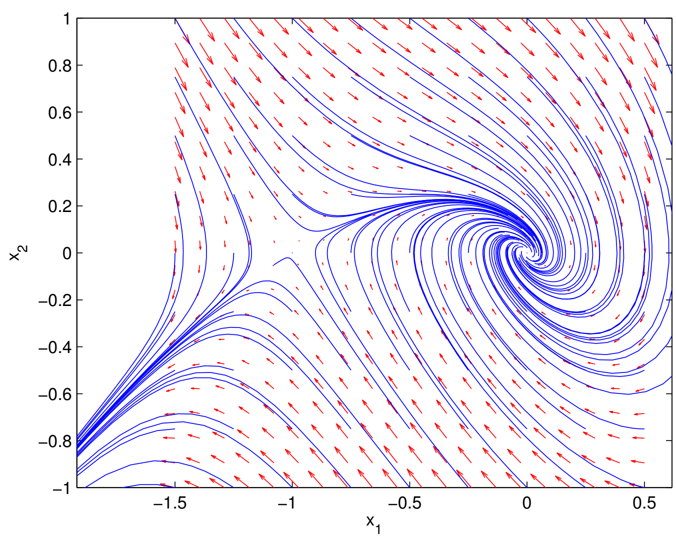

Figure 1: Phase plot of the 2-d system showing the two equilibria: left is a saddle point, right is a stable focus, there is a separatrix between them.

Example 2. *Third-order system.* Consider the system

$$\dot{x} = ax^3$$

Linearizing about the origin we have

$$A = \partial f|_{x=0} = 3ax^2|_{x=0} = 0,$$

There is one eigenvalue which lies on the imaginary axis, so we cannot conclude stability using linearization. If $a < 0$ the origin is AS considering the Lyapunov function

$$V(x) = x^4,$$

whose derivative $\dot{V}(x) = 4ax^6 < 0$ for all $x \neq 0$. If $a = 0$ the system is linear and the origin is AS. If $a > 0$ the origin is unstable since $\dot{V} = 4ax^6$. Note that we could have also shown AS using a Lyapunov function $V(x) = x^2$.

Example 3. *More complex Lyapunov function.*Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 - 2x_2 + 2x_1^3\end{aligned}$$

and the Lyapunov function $V(x) = x_1^2 + x_2^2 + x_1^4$. We have

$$\begin{aligned}\dot{V} &= 2x_1x_2 - 2x_1^4 - 2x_1x_2 - 4x_2^2 + 4x_1^3x_2 + 4x_1^3x_2 - 4x_1^6 \\ &= -4(x_1^3 - x_2)^2 - 2x_1^4,\end{aligned}$$

which shows that the system is asymptotically stable. Furthermore, V is radially unbounded which implies global stability.

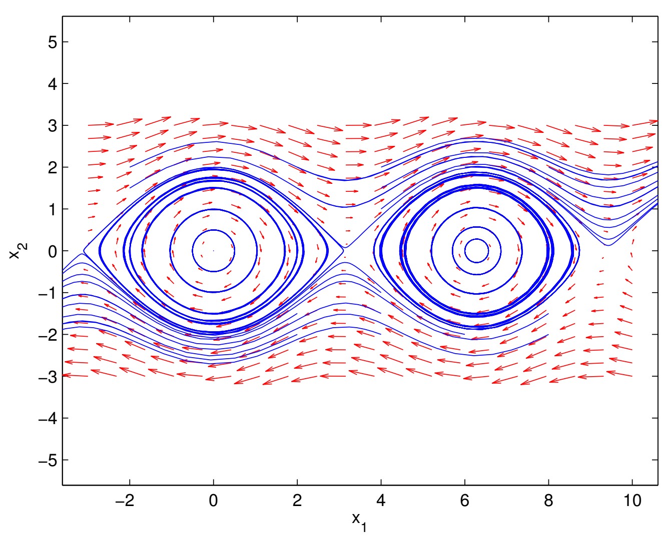

Figure 2: Phase plot of the pendulum without damping showing several equilibria: the "eyes" at $\theta = 0 \pm 2\pi k$ are stable centre, the other at $\theta = \pi + \pm 2\pi k$ are saddles.

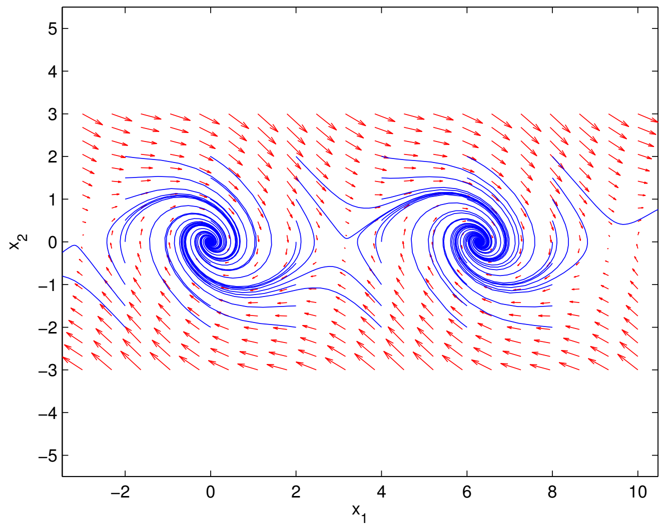

Figure 3: Phase plot of the pendulum with damping showing several equilibria: the "eddies" at $\theta = 0 \pm 2\pi k$ are stable foci, the other at $\theta = \pi + \pm 2\pi k$ are saddles.

Example 4. *Example: pendulum* Dynamics: $\ddot{\theta} + B\dot{\theta} + \sin\theta = 0$, for $0 < B < 2$.

Lyapunov's first method: set $x_1 = \theta$, $x_2 = \dot{x}_1$,

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = -\sin x_1 - Bx_2 \tag{4}$$

- critical points: $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, ...$
- at even n we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & B \end{pmatrix}, \lambda_{1,2} = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - 1} \quad \Rightarrow \text{ stable foci (if } B \neq 0\text{), otherwise undetermined}$$

- at odd n we have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -B \end{pmatrix}, \lambda_{1,2} = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} + 1} \quad \Rightarrow \text{ saddle points}$$

Consider the pendulum without damping, i.e. $B = 0$ (Figure **??**). Now consider adding damping, e.g. $B = 0.5$ (Figure **??**).

Lyapunov's second method Consider the function $V = \frac{1}{2}\dot{\theta}^2 + (1 - \cos\theta)$ which actually corresponds to the system total energy. Consider the equilibrium $\theta^* = (0, 0)$. The function is locally p.d. around θ^* for $|\theta| < \pi$. Furthermore, it is *not* radially unbounded. Thus, the analysis will be only local. We have

$$\dot{V} = \dot{\theta}\ddot{\theta} + \dot{\theta}\sin\theta = -B\dot{\theta}^2 \leq 0$$

so $\dot{V}$ is n.s.d. implying that the system is stable, but not necessarily asymptotically stable. In particular, we have $\dot{V} = 0$ for $\dot{\theta} = 0$ and *any* θ.

Nevertheless, notice that $\dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\sin\theta$, then $\dot{\theta}$ will change if $\theta \neq 0$ so that θ will also tend to zero.

A generalization known as LaSalle invariance principle then deduces asymptotic stability: *Let S be all points x for which $\dot{V} = 0$. If no solution can stay in S other than $x = 0$ then the system is asymptotically stable.*

In the pendulum example we weren't able to show asymptotic stability because $\dot{V}$ is n.s.d., i.e. we have $\dot{V} = 0$ at some points different than $x = 0$. LaSalle's principle implies asymptotic stability based on the following idea: if the system starts at $x(0)$ such that $\dot{V}(x(0)) = 0$, it will immediately leave the set $\{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$ and come back to it only at $x = 0$. But in case when $\dot{V} = 0$ persists along the solution, then the system is not asymptotically stable.

We formalize this as follows:

Definition 4. Invariant set *([?]) The set $M \subset R^n$ is said to be a (positively)* invariant set *if for all $y \in M$ and $t_0 \geq 0$, we have*

$$s(t, y, t_0) \in M, \quad \forall t \geq 0.$$

In other words, if a state originates in an invariant set, it remains there.

Theorem 4. Lasalle's principle *([?]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ we have $\dot{V}(x) \leq 0$. Define*

$$S = \{x \in \Omega_c \ : \ \dot{V}(x) = 0\}.$$

As $t \to \infty$ the trajectory tends to the largest invariant set inside S. In particular, if S contains no invariant sets other than $x = 0$, then the origin is asymptotically stable. Equivalently,

- *If no solution can stay identically in S, other than the trivial solution $x(t) = 0$, then the origin is asymtptically stable*
- *If $\Omega_c \subset \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is* globally *asymptotically stable*

Example 5. *Pendulum.* Continuing the pendulum example we have

$$S = \{(x_1, x_2 = 0)\}$$

the system will stay in S only when $\dot{x}_2 = 0$ which means $\sin x_1 = 0$ or that $x_1 = k\pi$ for any integer k. If in the region $x_1 \in (-\pi, \pi)$ the system will maintain $\dot{V} = 0$ in S only when $x_1 = 0$. Thus, the system is locally asymptotically stable at the origin.

We can also construct the set Ω_c by choosing $c = V((\pm\pi, 0)) = 2$ which physically means starting with zero velocity infinitely close to the vertical position. Thus, all points in the set

$$\Omega_2 = \{(\theta, \dot{\theta}) \mid \frac{1}{2}\dot{\theta}^2 + 1 - \cos\theta < 2\}$$

will asymptotically stabilize to 0.

Example 6. *Linear damped harmonic oscillator.* Consider $x = (q, \dot{q})$, $M, B, K > 0$

$$M\ddot{q} + B\dot{q} + Kq = 0$$

Construct Lyapunov function:

$$V = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Kq^2 = \frac{1}{2}x^T \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} x$$
$$\dot{V} = M\ddot{q}\dot{q} + K\dot{q}q = -B\dot{q}^2 \quad \Rightarrow \quad \text{stable}$$

We have $S = \{(q, 0)\}$. Invariance of S requires that $\ddot{q} = 0 \Rightarrow$ which requires that $q = 0$. Since $(0, 0)$ is the only invariant set within S then the system is asymptotically stable. Furthermore, V is radially unbounded which implies global stability.

In this particular example, we could have shown asymptotic (in fact even exponential) stability without using LaSalle, but through a different cost function:

$$V = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Kq^2 + \epsilon\dot{q}Mq = \frac{1}{2}x^T \begin{pmatrix} K & \epsilon M \\ \epsilon M & M \end{pmatrix} x = x^T P x$$
$$\text{verify that } \dot{V} = -x^T \begin{pmatrix} \epsilon K & \frac{\epsilon}{2}B \\ \frac{\epsilon}{2}B & B - \epsilon M \end{pmatrix} x = -x^T D x,$$

where D is p.d. for a small enough ϵ.

0.3.3 The Linear Case

Consider a linear system

$$\dot{x} = Ax, \quad V(x) = x^T P x, P = P^T > 0,$$

We have

$$\dot{V}(x) = x^T P\dot{x} + \dot{x}Px = x^T(PA + A^T P)x \triangleq -x^T Q x$$

If $Q > 0$ then A is Hurwitz (since the system must be asymptotically stable).

Alternatively, choose $Q > 0$ and solve the Lyapunov equation for P

$$PA + A^T P = -Q,$$

if $P > 0$ then A is Hurwitz. This is done with $P = \texttt{lyap}(A^T, Q)$.

Theorem 5. *A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is a $P = P^T > 0$ that satisfies the Lyapunov equation*

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz then P is the unique solution.

Theorem 6. Exponential Stability Theorem. *The point $x_0 = 0$ is an exponentially stable equilibrium of $\dot{x} = f(t, x)$ iff there exists an $\epsilon > 0$ and a function $V(t, x)$ that satisfies*

$$\alpha_1\|x\|^2 \leq V(t, x) \leq \alpha_2\|x\|^2 \tag{5}$$
$$\dot{V} \leq -\alpha_3\|x\|^2 \tag{6}$$
$$\|\frac{\partial V}{\partial x}(t, x)\| \leq \alpha_4\|x\|, \tag{7}$$

for some positive constants α_i and $\|x\| \leq \epsilon$. The rate of convergence is then determined by

$$m \leq \sqrt{\frac{\alpha_2}{\alpha_1}}, \qquad \alpha = \frac{\alpha_3}{2\alpha_2}. \tag{8}$$

Proof:

$$\alpha_1\|x\|^2 \leq V(t,x) \leq \alpha_2\|x\|^2 \tag{9}$$
$$\dot{V} \leq -\alpha_3\|x\|^2 \tag{10}$$
$$\leq -\frac{\alpha_3}{\alpha_2}V(x) \tag{11}$$

Therefore, we have

$$\begin{aligned}
&\dot{V}(x) \leq -\frac{\alpha_3}{\alpha_2}V(x) \\
&V(x) \leq V(x_0)e^{-\frac{\alpha_3}{\alpha_2}t} \\
&\|x\| \leq \left(\frac{V(x)}{\alpha_1}\right)^{\frac{1}{2}} \leq \left(\frac{V(x_0)e^{-\frac{\alpha_3}{\alpha_2}t}}{\alpha_1}\right)^{\frac{1}{2}} \\
&\|x(t)\| \leq \|x_0\|\sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\frac{\alpha_3}{2\alpha_2}t}
\end{aligned}$$

□

For quadratic Lyapunov functions $V = x^TPx$ with $\dot{V} = -x^TQx$, where $P, Q > 0$ we have

$$V(t) \leq e^{-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}t}V(0) \Rightarrow \text{ exponential energy decay}$$

which follows directly from the fact that for a p.d. martix P we have $\lambda_{min}(P)\|x\|^2 \leq x^TPx \leq \lambda_{max}(P)\|x\|^2$.

Example 7. *The nonlinear spring-damper [**?**].*Consider the dynamics

$$\dot{x}_1 = x_2, \tag{12}$$
$$\dot{x}_2 = -f(x_2) - g(x_1), \tag{13}$$

where $f(x)$ and $g(x)$ are nonlinear smooth functions modeling the friction in the damper and the restoring force in the sptring, respectively. We will assume that f and g are *passive*, i.e.

$$\sigma f(\sigma) \geq 0, \qquad \sigma g(\sigma) \geq 0, \text{ for all } \sigma \in [-\sigma_0, \sigma_0]$$

with equality only when $\sigma = 0$. The candidate for the Lyapunov function is

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)d\sigma$$

Passivity implies that $V(x)$ is a locally positive definite function. Then we have

$$\dot{V}(x) = -x_2 f(x_2) \leq 0,$$

for $|x_2| \leq \sigma_0$ which shows that the system is stable but not necessarily asymptotically stable. We apply LaSalle's principle by considering the Lyapunov function value $c = \min(V(\sigma_0, 0), V(-\sigma_0, 0))$ for which we can see that

$$\dot{V}(x) \leq 0 \text{ for all } x \in \Omega_c \triangleq \{x : V(x) \leq c\}$$

Consider the region

$$S = \Omega_c \cap \{x_1, x_2 | \dot{V} = 0)\} = \Omega_c \cap \{(x_1, 0)\},$$

to obtain the largest invariant set within S note that

$$x_2(t) = 0 \Rightarrow x_1(t) = x_{10} \Rightarrow \dot{x}_2 = 0 = -f(0) - g(x_{10}),$$

where x_{10} is some constant. This means that $g(x_{10}) = 0$ or that $x_{10} = 0$. Which means that the largest invariance set in S is the origin, i.e. that the system is asymptotically stable.

0.4 Using feedback to design stabilizing control

Consider systems of the form $\dot{x} = f(x, u)$. We will not investigate in-depth topics such as Input-to-State-Stability (ISS) and Input-Output Stability (IOS). Instead we will study how control is used to obtain desired stability as pertitent to applications in robotics.

At a basic level, our goal is to obtain u in a feedback-form, i.e.

$$u = \phi(x),$$

so that the resulting *closed-loop* systems has the dynamics

$$\dot{x} = f(x, \phi(x))$$

Example 8. *1-d examples.* Consider the system

$$\dot{x} = ax^2 - x^3 + u, \text{ for some } a \neq 0$$

The simplest approach is to set

$$u = -ax^2 + x^3 - x$$

which results in the closed-loop system

$$\dot{x} = -x$$

which is exponentially stable. This approach was to simply cancel all nonlinear terms. But actually, it is not really necessary to cancel the term $-x^3$ since it is already dissipative. A more economical control law would have just been:

$$u = -ax^2 - x$$

The question of determining a proper u also comes down to finding a Lyapunov function. One approach is to actually specify the Lyapunov function V and a negative definite $\dot{V}$ and then find u to match these choices. For instance, in the example above, let

$$V(x) = \frac{1}{2}x^2$$

and let

$$\dot{V} = ax^3 - x^4 + xu \leq -L(x),$$

for some positive definite $L(x)$. One choice is $L(x) = x^2$ which results in

$$u = -ax^2 + x^3 - x,$$

i.e. the same expensive control law. But another choice is to include higher-order terms, i.e. $L(x) = x^2 + x^4$. Then we have

$$u = -ax^2 - x,$$

which is the preffered control law to globally asymptotically stabilize the system.

Next consider the the trajectory tracking of standard fully-actuated robotic systems. The dynamics is given by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q, \dot{q}) = u$$

and the task is to track a desired trajectory $q_d(t)$ which is at least twice differentiable. The computed torque law is given by

$$u = M(q)(\ddot{q}_d - K_d\dot{e} - K_pe) + C(q, \dot{q})\dot{q} + N(q, \dot{q}),$$

where $e = q - q_d$ and K_p and K_d are constant matrices. When we substitute this control law we have the following error dynamics

$$\ddot{e} + K_d\dot{e} + K_pe = 0.$$

Since this is a linear equation it is easy to choose K_d and K_p to guarantee that the system is exponentially stable.

Theorem 7. *Stability of computed torque law.* If $K_p, K_d \in \mathbb{R}^{n\times n}$ are positive definite symmetric matrices, then the computed torque law results in exponential trajectory tracking.
Proof: We have the dynamics

$$\frac{d}{dt}\begin{pmatrix} e \\ \dot{e} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{\triangleq A} \begin{pmatrix} e \\ \dot{e} \end{pmatrix}$$

We can show that the eigenvalues of A have negative real parts. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $v = (v_1, v_2) \in \mathbb{C}^{2n}$, $v \neq 0$. Then

$$\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{\triangleq A} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -K_pv_1 - K_vv_2 \end{pmatrix},$$

which means that if $\lambda = 0$ then $v = 0$ and so $\lambda = 0$ is not an eigenvalue. Similarly, $v_1, v_2 \neq 0$ and we may assume that $\|v_1\| = 1$. Then we have

$$\begin{aligned} \lambda^2 &= v_1^*\lambda^2 v_1 = v_1^*\lambda v_2 \\ &= v_1^*(-K_pv_1 - K_dv_2) = -v_1^*K_pv_1 - \lambda v_1^*K_dv_1, \end{aligned}$$

where $*$ denotes complex conjugate transpose. Since $\alpha \triangleq v_1^*K_pv_1 > 0$ and $\beta \triangleq v_1^*K_dv_1 > 0$ we have

$$\lambda^2 + \alpha\lambda + \beta = 0, \qquad \alpha, \beta > 0,$$

the real part of λ must be negative. □

This is an example of a more general technique known as *feedback linearization*. In subsequent lectures we will generalize these results to underactuated or constrained systems.