EN530.678 Nonlinear Control and Planning in Robotics
Lecture 2: System Models
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0.1 Constraints

The configuration space of a mechanical sysetm is denoted by Q and is assumed to be an n-dimensional manifold, locally isomorphic to $\mathbb{R}^n$ (we'll say exactly what this means in a future lecture). A configuration is denoted by $q \in Q$.

We first introduce the notion of constraints:

- holonomic (or geometric):
$$h_i(q) = 0, \quad i = 1, \ldots, k$$
restrict possible motions to a $n - k$ dimensional sub-manifold (think hypersurface embedded in a larger ambient space)
- linear (Pfaffian) nonholonomic (or kinematic):
$$a_i^T(q)\dot{q} = 0, \quad i = 1, \ldots, k, \qquad \text{or} \qquad A^T(q)\dot{q} = 0 \quad \text{in matrix form}$$
linear in the velocities

Nonholonomic constraints are not integrable, i.e. it is not possible to find k functions h_i such that
$$\nabla_q h_i(q) = a_i(q), \quad i = 1, \ldots, k$$
If one can find such functions then the constraint is holonomic, i.e.
$$\int a_i^T(q(t))\dot{q}(t)dt = \int \nabla h_i(q(t))^T \dot{q}(t)dt = h_i(q) + c,$$
where c is a constant of integration.

Holonomic constraints are inherently different than nonholonomic. If $a(q)^T\dot{q} = 0$ can be integrated to obtain $h(q) = c$, then the motion is restricted to lie on a level surface (a leaf) of h corresponding to the constant c obtained by the initial condition $c = h(q_0)$. Practically speaking, once the system is on the surface, it cannot escape.

Consider a single constraint $a(q)^T\dot{q} = 0$. When the constraint is nonholonomic the *instantaneous motion* (velocity) is allowed in all directions except for $a(q)$ (i.e. to an $n-1$-dimensional space). But it could still be *possible to reach any configuration* in Q. So the system will leave the surface.

Example 1. *The unicycle.* The canonical example of a nonholonomic system is the unicycle (a.k.a. the rolling disk). The configuration is $q = (x, y, \theta)$ denoting position (x, y) and orientation θ. There is one constraint, i.e. the unicycle must move in the direction in which it is pointing:
$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0, \qquad \text{or} \qquad \frac{\dot{y}}{\dot{x}} = \tan\theta,$$

We have

$$a(q) = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}.$$

The feasible velocities are then contained in the null space of $A(q) = a(q)$, i.e.

$$\text{null}(a^T(q)) = \text{span}\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This system starts at configuration $q_0 = (x_0, y_0, \theta_0)$ and can reach any desired final configuration $q_f = (x_f, y_f, \theta_f)$. The simplest strategy is first to rotate so that the disk points to (x_f, y_f), then move forward until (x_f, y_f) is reached, then turn in place until the orientation reaches θ_f.

Draw a picture of the motion in the the configuration space.

More generally, let us denote the *allowed directions* of motion by vectors g_j, i.e.

$$a_i(q)^T g_j(q) = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n-k$$

or in matrix form

$$A^T(q)G(q) = 0.$$

The feasible trajectories of the mechanical system are the solutions of

$$\dot{q} = \sum_{j=1}^{m} g_j(q) v_j = G(q)v,$$

where $v(t) \in \mathbb{R}^m$, $m = n - k$, are called reduced velocities or *pseudovelocities*

We will be concerned with two classes of models. *Kinematic* models assume that v can be directly controlled. Dynamic models require the derviation of another differential equation determining the evolution of v.

For *kinematic* systems the question of controllability is equivalent to nonholonomy.

0.2 Dynamics

How do we obtain $\dot{x} = f(t, x, u)$ for dynamical systems? We will focus on mechanical systems with equations of motion derived through a Lagrangian approach, which is often sufficient for most systems of interest in robotics.

0.2.1 Holonomic Underactuated Systems

Let $q \in \mathbb{R}^n$ denote generalized coordinates. Assume that the system has a Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q),$$

with inertia matrix $M(q) > 0$ and potential energy $V(q)$. The system is subject to external forces $f_{\text{ext}}(q, \dot{q}) \in \mathbb{R}^n$ and control inputs $u \in \mathbb{R}^m$.

The equations of motion in terms of the Lagrangian (i.e.the Euler-Lagrange equations) are given by

$$\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L = f_{\text{ext}}(q,\dot{q}) + B(q)u,$$

where $B(q) \in \mathbb{R}^{n\times m}$ is a matrix mapping from m control inputs to the forces/torques acting on the generalized coordinates q.

This equation is obtained from Lagrange-d'Alembert variational principle

$$\delta\int_{t_0}^{t_f} L(q,\dot{q})dt + \int_{t_0}^{t_f}[f_{\text{ext}}(q,\dot{q}) + B(q)u)]^T\delta q(t) = 0.$$

The actual equations take the form

$$M(q)\ddot{q} + b(q,\dot{q}) = B(q)u, \tag{1}$$

where

$$b(q,\dot{q}) = \dot{M}(q)\dot{q} - \frac{1}{2}\nabla_q(\dot{q}^T M(q)\dot{q}) + \nabla_q V(q) - f_{\text{ext}}(q,\dot{q}).$$

The system is written in control form in terms of the state $x = (q,\dot{q})$ as

$$\dot{x} = f(x) + g(x)u = \begin{pmatrix} \dot{q} \\ -M(q)^{-1}b(q,\dot{q}) \end{pmatrix} + \begin{pmatrix} 0 \\ M(q)^{-1}B(q) \end{pmatrix} u$$

Example 2. *2-dof manipulator.* Consider a 2 dof-manipulator subject to gravity with the following parameters:

Description	Notation
Length of link #1	l_1
Length of link #2	l_2
Distance to COM of link #1	l_{c1}
Distance to COM of link #2	l_{c2}
link #1 mass	m_1
link #2 mass	m_2
link #1 inertia	I_1
link #2 inertia	I_2
gravity acceleration	g

The mass matrix is

$$M(q) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2[l_1^2 + l_{2c}^2 + 2l_1l_{c2}\cos q_2] + I_1 + I_2 & m_2(l_{c2}^2 + l_1l_{c2}\cos q_2) + I_2 \\ m_2(l_{c2}^2 + l_1l_{c2}\cos q_2) + I_2 & m_2l_{c2}^2 + I_2 \end{bmatrix},$$

while the bias term is

$$b(q,\dot{q}) = \begin{bmatrix} -m_2l_1l_{c2}\sin(q_2)\dot{q}_2 & -m_2l_1l_{c2}\sin(q_2)[\dot{q}_1+\dot{q}_2] \\ m_2l_1l_{c2}\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}\dot{q} + \begin{pmatrix} [m_1l_{c1} + m_2l_1]g\sin(q_1) + m_2l_{c2}g\sin(q_1+q_2) \\ m_2l_{c2}g\sin(q_1+q_2) \end{pmatrix},$$

For fully actuated manipulator we have $B(q) = I$. For actuation only at the first joint we have

$$B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 3. *Simplified model of a boat in 2D, with two rear propellers.* The configuration is denoted by $q = (x, y, \theta)$. The mass matrix is given by

$$M(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}.$$

while the bias is

$$b(q, \dot{q}) = R(\theta)D(\dot{q})R(\theta)^T\dot{q},$$

where tha matrix $D(\dot{q}) \geq 0$ denotes drag terms and $R(\theta)$ is the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which transforms forces from body-fixed to spatial frame. The control martix is

$$B(q) = R(\theta) \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -r & r \end{bmatrix},$$

where the constant $r > 0$ denotes the distance between each thruster and central axis.

0.2.2 Nonholonomic Systems

Assume that the system has a Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T K(q)\dot{q} - V(q),$$

with inertia matrix $K(q) > 0$ and potential energy $V(q)$. The system is subject to external forces $f_{\text{ext}}(q, \dot{q})$ and control inputs $u \in \mathbb{R}^m$.

The Euler-Lagrange equations take the form

$$\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L = A(q)\lambda + f_{\text{ext}}(q, \dot{q}) + S(q)u,$$

where $S(q) \in \mathbb{R}^{n \times m}$ is a matrix mapping from m control inputs to the forces/torques acting on the generalized coordinates q and where $\lambda \in \mathbb{R}^k$ is a vector of Lagrange multipliers. The term $A(q)\lambda$ should be understood as a force which counters any motion in directions spanned by $A(q)$.

This equation is obtained from the Lagrange-d'Alembert variational principle

$$\delta \int_{t_0}^{t_f} L(q, \dot{q})dt + \int_{t_0}^{t_f} [f_{\text{ext}}(q, \dot{q}) + S(q)u)]^T \delta q(t) = 0,$$

subject to *both* $A(q)^T\dot{q} = 0$ *and* $A(q)^T\delta q(t) = 0$, i.e. the variations are restricted as well.

The actual equations take the form

$$K(q)\ddot{q} + n(q, \dot{q}) = A(q)\lambda + S(q)u, \tag{2}$$
$$A^T(q)\dot{q} = 0, \tag{3}$$

where

$$n(q,\dot q) = \dot K(q)\dot q - \frac{1}{2}\nabla_q(\dot q^T K(q)\dot q) + \nabla_q V(q)$$

The Lagrange multipliers can be eliminated by first noting that

$$A^T(q)G(q) = 0$$

and multiplying (2) by $G^T(q)$ to obtain a reduced set of $m = n - k$ differential equations

$$G^T(q)(K(q)\ddot q + n(q,\dot q)) = G^T S(q)u.$$

A standard assumption will be that $\det(G(q)^T S(q)) \neq 0$ or that all feasible directions are controllable. The final equations are then expressed as

$$\dot q = G(q)v, \tag{4}$$

$$M(q)\dot v + b(q,v) = B(q)u, \tag{5}$$

where

$$\begin{aligned} M(q) &= G^T(q)K(q)G(q) > 0 \\ b(q,v) &= G^T K(q)\dot G(q)v + G^T(q)n(q,G(q)v) \\ B(q) &= G^T(q)S(q) \end{aligned}$$

using the notation

$$\dot G(q)v = \sum_{i=1}^{m}(\nabla g_i(q)^T v_i)G(q)v.$$

For nonholonomic systems, we would normally assume an isomorphism between *pseudo-accelerations* $a = \dot v$ and control inputs u, i.e. any acceleration a can be achieved by setting

$$u = B(q)^{-1}(M(q)a + b(q,v)).$$

That is why often in nonholonomic control we take a as the (virtual) control input, i.e. $u \equiv a$ and express the control system in terms of the state $x = (q,v)$

$$\dot x = f(x) + g(x)u = \begin{pmatrix} G(q)v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u.$$

Example 4. *Unicycle.* The configuration is $q = (x, y, \theta)$ with mas m, moment of inertia J, driving force u_1, steering force u_2. The general dynamic model

$$K(q)\ddot q + n(q,\dot q) = A(q)\lambda + S(q)u,$$

takes the form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \begin{pmatrix} \ddot x \\ \ddot y \\ \ddot\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}\lambda + \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

We have $G(q) = S(q)$, $G^T(q)S(q) = I_2$, and $G^T(q)B\dot{G}(q) = 0$, from which we obtain the reduced mass matrix and bias

$$M(q) = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}, \qquad b(q, \dot{q}) = 0.$$

The complete equations of motion are

$$\begin{aligned} \dot{x} &= \cos\theta v_1 \\ \dot{y} &= \sin\theta v_1 \\ \dot{\theta} &= v_2 \\ m\dot{v}_1 &= u_1 \\ J\dot{v}_2 &= u_2, \end{aligned}$$

which can be put in a standard form, for $\mathtt{x} = (x, y, \theta, v_1, v_2)$

$$\dot{\mathtt{x}} = f(\mathtt{x}) + g(\mathtt{x})u.$$

Example 5. *Simple car models.* A common way to model a car for control purposes is to employ the bycycle model, i.e. collapse each pair of wheels to a single wheel at the center of their axle. The generalized coordinates are

$$q = (x, y, \theta, \phi),$$

where ϕ is the *steering angle*. We have the constraints

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \qquad \text{front wheel} \tag{6}$$

$$\dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) - \dot{\theta}\ell\cos\phi = 0 \qquad \text{rear wheel} \tag{7}$$

For the real-wheel drive we have

$$G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{\ell}\tan\phi & 0 \\ 0 & 1 \end{bmatrix}$$

while for the front-wheel drive we have

$$G(q) = \begin{bmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ \frac{1}{\ell}\sin\phi & 0 \\ 0 & 1 \end{bmatrix}$$

Dynamic vs kinematic model. A *kinematic model* is given by

$$\dot{q} = G(q)u,$$

where the inputs $u \in \mathbb{R}^m$ are actually the pseudo-velocities (that we defined above as $v \in \mathbb{R}^m$), i.e. u_1 – rear drive velocity, u_2 - steering rate. A dynamic model includes the dynamics of $\dot{v}$ and the control inputs u are forces or accelerations (e.g. similar to the unicycle).