

EN530.678 Nonlinear Control and Planning in Robotics
Lecture 10: Lyapunov Redesign and Robust Backstepping
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1 Uncertainty and Lyapunov Redesign

Consider the system [?]

$$\dot{x} = f(t, x) + G(t, x)[u + \delta(t, x, u)], \quad (1)$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^p$ is the control input. The functions f, G , and δ are defined for $(x, u) \in D \times \mathbb{R}^p$, where $D \subset \mathbb{R}^n$ contains the origin. The functions f, G and δ are piece-wise continuous and Lipschitz in x and u . We assume that f and G are known while δ is unknown and represents the combined effect of *model simplification, parametric uncertainty*, etc... [?]. When the uncertainty acts only along control vector fields (the columns of the matrix G) it is said to satisfy the *matching condition*, i.e. it matches the controls. The system (1) is in such form. Stabilizing controls can be designed for this case through the concept of *Lyapunov redesign*. In the non-matching case, it is necessary to assume more restrictive assumptions about the bounds of δ and employ recursive techniques such as *robust backstepping*.

A nominal model of the system is given by

$$\dot{x} = f(t, x) + G(t, x)u, \quad (2)$$

and we assume that a feedback controller $u = \psi(t, x)$ was designed so that the nominal closed-loop system

$$\dot{x} = f(t, x) + G(t, x)\psi(t, x), \quad (3)$$

is uniformly asymptotically stable.

Assume that the nominal control corresponds to a Lyapunov function $V(t, x)$ such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \quad (4)$$

$$\partial_t V + \partial_x V \cdot [f(t, x) + G(t, x)\psi(t, x)] \leq -\alpha_3(\|x\|), \quad (5)$$

for all $x \in D$ and where the functions α_i are strictly increasing and satisfy $\alpha_i(0) = 0$ (such functions are said to be class \mathcal{K} functions). Assume that for

$$u = \psi(x, t) + v,$$

the uncertainty satisfies the bound

$$\|\delta(t, x, \psi(t, x) + v)\| \leq \rho(t, x) + k_0\|v\|, \quad 0 \leq k_0 < 1 \quad (6)$$

where $\rho : [0, t_f] \times D \rightarrow \mathbb{R}$ is a non-negative continuous function and specifies the magnitude of the uncertainty. The idea behind Lyapunov redesign is to augment the nominal control law $\psi(t, x)$ with an extra term $v \in \mathbb{R}^p$ which suppresses the uncertainty so that the combined control $u = \psi(t, x) + v$ stabilizes the real system (1).

The closed-loop system now becomes

$$\dot{x} = f(t, x) + G(t, x)\psi(t, x) + G(t, x)[v + \delta(t, x, \psi(t, x) + v)]. \quad (7)$$

The derivative of V is computed becomes

$$\dot{V} = \partial_t V + \partial_x V \cdot [f + G\psi] + \partial_x V \cdot G[v + \delta] \leq -\alpha_3(\|x\|) + \partial_x V \cdot G[v + \delta] \quad (8)$$

Setting $w^T = \partial_x V \cdot G$ this is equivalent to

$$\dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \delta \quad (9)$$

Using the bound (6) we have

$$w^T v + w^T \delta \leq w^T v + \|w\| (\rho + k_0 \|v\|) \quad (10)$$

Setting

$$v = -\eta(t, x) \frac{w}{\|w\|}, \quad (11)$$

for some $\eta(t, x) > 0$ such that

$$\eta(t, x) \geq \frac{\rho(t, x)}{1 - k_0}, \quad \forall x \in D$$

we have

$$w^T v + w^T \delta \leq -\eta(x)\|w\| + \|w\|(\rho + k_0\eta(x)) = \|w\|(\rho - \eta(1 - k_0)) \leq 0$$

Hence, $\dot{V} \leq 0$ for the whole system.

Note that the uncertainty bound (6) was employed by regarding the norm $\|\cdot\|$ as a L_2 norm $\|\cdot\|_2$. An alternative controller can be obtained by setting $\|\cdot\| = \|\cdot\|_\infty$ (see [?]).

The resulting controller (11) is discontinuous at $w = 0$, e.g. typically at the origin. In addition to this theoretical limitation, practical issues also occur due to digital switching, delays, and other physical imperfections. This results in oscillatory behavior near the equilibrium called *chattering*. In order to deal with it the control law can be *smoothed* near the origin by setting

$$v = -\eta(t, x) \frac{w}{\|w\|}, \quad \text{if } \eta(t, x)\|w\| \geq \epsilon, \quad (12)$$

$$v = -\eta(t, x)^2 \frac{w}{\epsilon}, \quad \text{if } \eta(t, x)\|w\| < \epsilon, \quad (13)$$

As a result one can show [?] that the closed-loop solutions of the system are bounded by a \mathcal{K} -class function of ϵ . Thus, by making ϵ arbitrary small the system can stabilize arbitrary close to the origin.

Example 1. Pendulum with uncertain model. Consider the pendulum with uncertain damping and control given by:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= a \sin x_1 + bx_2 + cu,\end{aligned}$$

where a and c are uncertain. By uncertain in this case we mean that their exact values are not known, but we do know that they are close to given *nominal* values, denoted by \hat{a} and \hat{c} . A stabilizing controller for the nominal system is chosen as

$$\psi(x) = -\frac{\hat{a}}{\hat{c}} \sin x_1 - \frac{1}{\hat{c}}(k_1 x_1 + k_2 x_2),$$

with gains $k_1 > 0$ and $k_2 > b$ so that the nominal system is asymptotically stable. The system is put in the form (1)

$$\dot{x}_2 = \hat{a} \sin x_1 + bx_2 + \hat{c}[u + \delta],$$

where

$$\begin{aligned}\delta &= \frac{a - \hat{a}}{\hat{c}} \sin x_1 + \frac{c - \hat{c}}{\hat{c}} u \\ &= (\bar{a} - \bar{c}) \frac{\hat{a}}{\hat{c}} \sin x_1 - \frac{\bar{c}}{\hat{c}} k^T x + \bar{c} v\end{aligned}$$

where we substituted $u = \psi(x) + v$ and used the notation

$$\bar{a} = \frac{a - \hat{a}}{\hat{a}}, \quad \bar{c} = \frac{c - \hat{c}}{\hat{c}}.$$

Hence, using the identities $\sin(x_1) \leq |x_1| \leq \|x\|$ and $k^T x \leq \|x\| \|k\|$ where $k = (k_1, k_2)$, the uncertainty can be expressed as

$$|\delta| \leq \rho_1 \|x\| + k_0 |v|,$$

where ρ_1 and k_0 must be chosen so that

$$\rho_1 \geq (|\bar{a}| + |\bar{c}|) \left| \frac{\hat{a}}{\hat{c}} \right| + \left| \frac{\bar{c}}{\hat{c}} \right| \|k\|, \quad k_0 \geq |\bar{c}|$$

In practice, we can make assumptions about how large $|\bar{a}|$ and $|\bar{c}|$ can be (e.g. less than 0.3, which would mean up to 30% error relative to the nominal value). From these assumptions we then set ρ_1 and k_0 .

2 Robust Backstepping

In the previous section we considered the case of uncertainty matched by the control inputs. This restriction can be relaxed by accounting for uncertainty in the context of backstepping. Consider the single-input system

$$\dot{\eta} = f(\eta) + g(\eta)\xi + \delta_\eta(\eta, \xi) \tag{14}$$

$$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta_\xi(\eta, \xi) \tag{15}$$

where $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}$ are defined over a domain $D \subset \mathbb{R}^{n+1}$ containing the origin $(0, 0)$. Assume that the functions f, g, f_a, g_a are smooth and known, while δ_η and δ_ξ are uncertain terms. In addition, it is assumed that f and f_a vanish at the origin and the uncertain terms satisfy

$$\|\delta_\eta(\eta, \xi)\|_2 \leq a_1 \|\eta\|_2 \quad (16)$$

$$|\delta_\xi(\eta, \xi)| \leq a_2 \|\eta\|_2 + a_3 |\xi|, \quad (17)$$

for all $(\eta, \xi) \in D$.

Assume that we can find a stabilizing controller $\xi = \phi(\eta)$ for $\phi(0) = 0$ for the system (14) and a Lyapunov function $V_0(\eta)$ such that

$$\frac{\partial V_0}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \xi)] \leq -b\|\eta\|^2 \quad (18)$$

for some $b > 0$. Suppose further that $\phi(\eta)$ satisfies

$$|\phi(\eta)| \leq a_4 \|\eta\|, \quad \left\| \frac{\partial \phi}{\partial \eta} \right\| \leq a_5 \quad (19)$$

over D . Consider the Lyapunov function

$$V(\eta, \xi) = V_0(\eta) + \frac{1}{2} [\xi - \phi(\eta)]^2$$

We have

$$\dot{V} = \frac{\partial V_0}{\partial \eta} [f + g\phi + \delta_\eta] + \frac{\partial V_0}{\partial \eta} g(\xi - \phi) + (\xi - \phi) \left[f_a + g_a u + \delta_\xi - \frac{\partial \phi}{\partial \eta} (f + g\xi + \delta_\eta) \right]$$

Taking

$$u = \frac{1}{g_a} \left[\frac{\partial \phi}{\partial \eta} (f + g\xi) - \frac{\partial V_0}{\partial \eta} g - f_a - k(\xi - \phi) \right], \quad k > 0 \quad (20)$$

we have

$$\dot{V} \leq -b\|\eta\|^2 + (\xi - \phi) \left[\delta_\xi - \frac{\partial \phi}{\partial \eta} \delta_\eta \right] - k(\xi - \phi)^2$$

Using assumptions (16),(17),(19) it can be shown that

$$\dot{V} \leq -b\|\eta\|^2 + 2a_6 |\xi - \phi| \|\eta\| - (k - a_3) |\xi - \phi|^2 \quad (21)$$

$$= - \begin{bmatrix} \|\eta\| \\ |\xi - \phi| \end{bmatrix} \begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix} \begin{bmatrix} \|\eta\| \\ |\xi - \phi| \end{bmatrix} \quad (22)$$

for some $a_6 > 0$. Choosing

$$k \geq a_3 + \frac{a_6^2}{b}$$

yields

$$\dot{V} \leq -\sigma [\|\eta\|^2 + |\xi - \phi|^2]$$

for some $\sigma > 0$.

Lemma 1. [?] Consider the system (14)-(15) where the uncertainty satisfies the inequalities (16),(17). Let $\phi(\eta)$ be a stabilizing state feedback control law for (14) that satisfies (19) and $V(\eta)$ a Lyapunov function that satisfies (18). Then, the state feedback control law (20) stabilizes the origin.