EN530.678 Nonlinear Control and Planning in Robotics
Homework #4

March 7, 2022

Due: March 14, 2022 (before class)

Prof: Marin Kobilarov

1. (10 pts) Consider a simple dynamic model of a car-like robot with state (x, y, θ, ϕ, v), where x, y denote the position, θ the orientation, and ϕ-angle of the steering wheel, and v is forward velocity:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v\cos\theta\cos\phi \\ v\sin\theta\cos\phi \\ v\sin\phi/\ell \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2, \tag{1}$$

where the constant $\ell > 0$ denotes the distance between the axles. Is the system: 1) LA, 2) controllable, 3) STLC? Show the necessary calculations to support your conclusion.

Hint: you can employ the definition of *dynamic extension*.

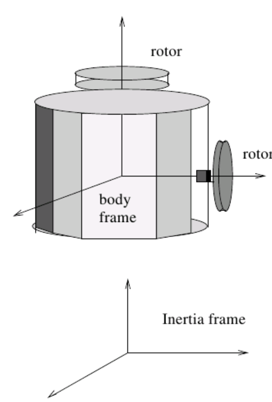

Figure 1: Satellite with two thrusters (MLS 7.12).

2. (20 pts) (adapted from MLS 7.12) Consider a model of a satellite body with two symmetrically attached rotors, where the rotors axes of rotation intersect at a point. The constraint on the system is conservation of angular momentum.

   (a) Assuming that the initial angular momentum of the system is zero, show that the (body) angular velocity, $\omega \in \mathbb{R}^3$, of the satellite body is related to the rotor velocities (u_1, u_2) by

$$\omega = b_1 u_1 + b_2 u_2$$

where $b_1, b_2 \in \mathbb{R}^3$ are constant vectors.

Hint: assume that the rotors are spinning around the x and y principle axes of rotation of the satellite, and denote the satellite body principle moments of inertia by $J = (J_1, J_2, J_3)$ while the rotor inertias by J_r. The Lagrangian is then $\ell(\omega) = \frac{1}{2}\omega^T J\omega + \frac{1}{2}J_r(\omega_1 + u_1)^2 + \frac{1}{2}J_r(\omega_2 + u_2)^2$. Now use the equations of motion $\frac{d}{dt}\partial_\omega \ell = \partial_\omega \ell \times \omega$ and the initial condition $\partial_w \ell(\omega(0)) = 0$.

(b) The equation above gives rise to a differential equation in the rotation group SO(3) for the satellite body

$$\dot{R}(t) = R(t)(\widehat{b}_1 u_1 + \widehat{b}_2 u_2),$$

where

$$\widehat{\omega} = \begin{bmatrix} 0 & -w_3 & w_3 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

Obtain a local coordinate description of this equation using the Euler parameters of SO(3) and show that the resulting system is controllable. In particular use XYZ angles:

$$R(\alpha, \beta, \gamma) = \begin{bmatrix} c_\beta c_\gamma & -c_\beta s_\gamma & s_\beta \\ c_\alpha s_\gamma + c_\gamma s_\alpha s_\beta & c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma & -c_\beta s_\alpha \\ s_\alpha s_\gamma - c_\alpha c_\gamma s_\beta & c_\gamma s_\alpha + c_\alpha s_\beta s_\gamma & c_\alpha c_\beta, \end{bmatrix},$$

with $c_\alpha \triangleq \cos\alpha$, $s_\alpha \triangleq \sin\alpha$, etc...

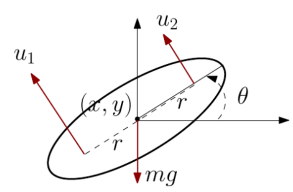

Figure 2: Underactuated rigid body

3. (10 pts) Read the paper "Differential Flatness of Mechanical Control Systems: A Catalog of Prototype Systems" (reference is also included in Lecture notes 7). The paper lists a system with *two colinear forces, off center* (Figure 2). Write down the equations of motion of this system. Show that the system is differentially flat by expressing the whole state and inputs as functions of the flat outputs and their derivatives.

4. (10 pts) Consider the control of a truck with a trailer (Figure 3) with equations of motion

$$\begin{aligned}
\dot{x} &= \cos\theta u_1 \\
\dot{y} &= \sin\theta u_1 \\
\dot{\theta} &= \frac{\tan\phi}{\ell} u_1 \\
\dot{\phi} &= u_2 \\
\dot{\theta}_1 &= \frac{1}{d}\sin(\theta - \theta_1) u_1,
\end{aligned}$$

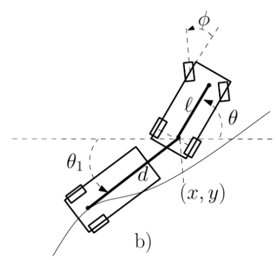

Figure 3: Truck towing a trailer

where (x, y, θ) are the position and orientation of the truck, ϕ is the steering wheels angle, θ_1 is the angle of the trailer, and ℓ and d are the lengths of the truck and the trailer.

Show that the system is differentially flat using the point between the rear wheels of the trailer as the flat output.