1. [Total 15 pts] Let $M$ be the ellipsoidal shell in $\mathbb{R}^3$ given by $x^2 + y^2 + 4z^2 = 1$. Show that $M$ is a manifold.

2. [Total 10 pts] Let $g_1$ and $g_2$ denote vector fields on $\mathbb{R}^3$ (with coordinates $(x, y, z)$) defined by

$$g_1 = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}, \quad g_2 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

(a) [5 pts] Show that $g_1$ and $g_2$ can actually be defined as vector fields on the standard two sphere $S^2$ of radius one.

(b) [5 pts] Calculate the Lie bracket $[g_1, g_2]$.

3. [Total 15 pts] Consider the distribution on $\mathbb{R}^3$ that is given at the point $(x, y, z) \in \mathbb{R}^3$ by the set of vectors $(a, b, c) \in \mathbb{R}^3$ satisfying $6ax + 2by + 10cz = 0$.

(a) [10 pts] Show that the distribution is integrable.

(b) [5 pts] Find the corresponding integrable manifolds defined by this distribution.

4. [Total 8 pts] A dynamical system in $\mathbb{R}^4$ can be described by

$$\dot{q} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} q_2 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} q_3 \\ 0 \\ 1 \\ 1 \end{pmatrix} u_3,$$

with input $u_1$, $u_2$, and $u_3$. Show it is nonholonomic.

5. [Total 10 pts] (MLS 7.2) Show that the differential constraint in $\mathbb{R}^5$ given by

$$(0, 1, \rho \sin q_5, \rho \cos q_3, \cos q_5)^T \dot{q} = 0,$$

for $q \in \mathbb{R}^5$ is nonholonomic.