EN530.678 Nonlinear Control and Planning in Robotics
Homework #2

February 16, 2022

Due: February 23, 2022 (before class)

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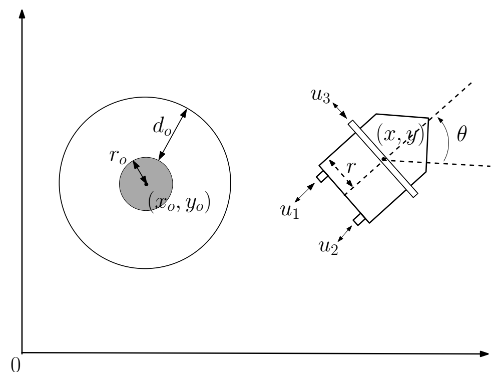

Figure 1: Omnidirectional hovercraft.

1. (Khalil) Consider the system

$$\dot{x}_1 = -x_2x_3 + 1, \quad \dot{x}_2 = x_1x_3 - x_2, \quad \dot{x}_3 = x_3^2(1 - x_3)$$

   (a) Show that the system has a unique equilibrium point. **(5 pts)**

   (b) Using linearization, show that the equilibrium point is asymptotically stable. Is it globally asymptotically stable? **(10 pts)**

2. (Khalil) Euler equations for a rotating rigid spacecraft are given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1,$$
$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2,$$
$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3,$$

   where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector ω along the principal axes, u_1, u_2, u_3 are the torque inputs applied about the principal axes, and J_1, J_2, J_3 are the principal moments of inertia.

   (a) **[3 pts]** Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable.

   (b) **[2 pts]** Is it asymptotically stable?

(c) **[5 pts]** Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_1, k_2, k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

3. (Khalil) Consider the m-link robot dynamics

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u,$$

where $q, u \in \mathbb{R}^n$, $M(q)$ is symmetric positive definite. The matrix C has the property that $\dot{M} - 2C$ is skew-symmetric [1] for all $q, \dot{q} \in \mathbb{R}^n$. The term $D\dot{q}$ accounts for viscous damping, where D is positive semidefinite symmetric matrix. The term $g(q)$ is computed according to $g(q) = \nabla P(q)$ where $P(q)$ is the potential energy of the system. Assume that $P(q) > 0$ $\forall q \neq 0$ and $g(q) = 0$ has an isolated root at $q = 0$.

(a) **[5 pts]** with $u = 0$ use the total energy $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as Lyapunov function to show that the origin $(q = 0, \dot{q} = 0)$ is stable.

(b) **[5 pts]** with $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) **[10 pts]** with $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^n , show that the point $(q = q^*, \dot{q} = 0)$ is an asymptotically stable equilibrium point.

4. Design of a stabilizing controller for a simple mechanical system and Matlab implementation.

Consider an omnidirectional hovercraft (Fig. 1) modeled as a fully actuated rigid body in the plane. It has mass m and moment of inertia J. It is controlled with three bidirectional thrusters. Two of them are placed in the rear at distance r from the central axis, and the third passes through the body laterally aligned with the center of mass. The hovercraft position is denoted by $p = (x, y)$ and its orientation by θ. The system coordinates are $q = (x, y, \theta)$. The forces produced by each thruster are denoted by $u = (u_1, u_2, u_3)$.

The equations of motion of the system can be expressed as

$$M\ddot{q} + D\dot{q} = B(q)u,$$

where

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix}, \; D = \begin{pmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_\theta \end{pmatrix}, \; B(q) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -r & r & 0 \end{pmatrix},$$

with $d_x, d_y, d_\theta > 0$ defining viscous damping constants.

Analytical Problems:

(a) **[10 pts]** Design an exponentially stable controller as a function of the state, i.e. $u = k(q,\dot{q})$, so that the system can stabilizes at the origin $(q,\dot{q}) = (0,0)$. Prove that your controller is exponentially stable.

[1] a skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$ has the property that $x^T S x = 0$ for all $x \in \mathbb{R}^n$

(b) **[10 pts]** Imagine that there is a disk-like obstacle at position $p_o = (x_o, y_o)$ with radius r_o that the vehicle must avoid. Assume that the vehicle can sense the obstacle if it is within d_o meters of it. Augment your control law with an obstacle avoidance term which applies a "steering" force to the (x, y) degrees of freedom defined by

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{k_o}{d(q)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

The force is applied only when the vehicle is heading towards an obstacle, i.e. when the angle between the velocity $(\dot{x}, \dot{y})$ and direction towards obstacle is less than $\pi/2$. Here, k_o is positive constant and $d(q) = \sqrt{(x - x_o)^2 + (y - y_o)^2} - r_o$ is the distance between the vehicle and obstacle. Prove the system is globally asymptotically stable.

Implementation:

Choose the following model parameters: $m = 1, J = .1, r = .2$, $D = \mathrm{diag}(.01, .1, .02)$.

(a) **[5 pts]** Obstacle-free case: implement the controller and simulate the closed-loop system from two initial conditions. In both cases set $q(0) = (3, 2, -\pi/4)$. The first initial condition must be with zero velocity (i.e. $\dot{q}(0) = 0$), while the second with non-zero velocity that you're free to choose.

(b) **[5 pts]** Obstacle avoidance case: add an obstacle with $r_0 = .25$ at position $p_o = (1, 1)$ and set $d_o = 1$. Design and simulate the obstacle avoidance controller from the two initial conditions specified in a). Generate trajectories for a few different choices of k_o and comment on the effect of this gain.

An example implementation of a simpler point-mass vehicle stabilization with obstacle avoidance is provided for reference. See file `hw2_example.m`.

Note: Upload your code and plots as a .zip file using `https://forms.gle/wDMCsuRfvczPGNDi6`; **in addition** attach a printout of the code and all plots to your homework solutions.

5. **[Extra Credit - 5 pts]**(Khalil) Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x,$$

where a is a positive constant, I_n is the nxn identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Note that this system is the same as in hw#1. Show that the origin is globally **exponentially** stable.