1. (Khalil) Consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_2 x_3 + 1, \\
\dot{x}_2 &= x_1 x_3 - x_2, \\
\dot{x}_3 &= x_2^3 (1 - x_3)
\end{align*}
\]

(a) Show that the system has a unique equilibrium point. (5 pts)
(b) Using linearization, show that the equilibrium point is asymptotically stable. Is it globally asymptotically stable? (10 pts)

2. (Khalil) Euler equations for a rotating rigid spacecraft are given by

\[
\begin{align*}
J_1 \dot{\omega}_1 &= (J_2 - J_3) \omega_2 \omega_3 + u_1, \\
J_2 \dot{\omega}_2 &= (J_3 - J_1) \omega_3 \omega_1 + u_2, \\
J_3 \dot{\omega}_3 &= (J_1 - J_2) \omega_1 \omega_2 + u_3,
\end{align*}
\]

where \( \omega_1, \omega_2, \omega_3 \) are the components of the angular velocity vector \( \omega \) along the principal axes, \( u_1, u_2, u_3 \) are the torque inputs applied about the principal axes, and \( J_1, J_2, J_3 \) are the principal moments of inertia.

(a) [3 pts] Show that with \( u_1 = u_2 = u_3 = 0 \) the origin \( \omega = 0 \) is stable.
(b) [2 pts] Is it asymptotically stable?
Suppose the torque inputs apply the feedback control \( u_i = -k_i \omega_i \), where \( k_1, k_2, k_3 \) are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

3. (Khalil) Consider the \( m \)-link robot dynamics

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D \dot{q} + g(q) = u,
\]

where \( q, u \in \mathbb{R}^n \), \( M(q) \) is symmetric positive definite. The matrix \( C \) has the property that \( \dot{M} - 2C \) is skew-symmetric\(^1\) for all \( q, \dot{q} \in \mathbb{R}^n \). The term \( D \dot{q} \) accounts for viscous damping, where \( D \) is positive semidefinite symmetric matrix. The term \( g(q) \) is computed according to \( g(q) = \nabla P(q) \) where \( P(q) \) is the potential energy of the system. Assume that \( P(q) > 0 \) for all \( q \neq 0 \) and \( g(q) = 0 \) has an isolated root at \( q = 0 \).

(a) [5 pts] with \( u = 0 \) use the total energy \( V(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} + P(q) \) as Lyapunov function to show that the origin \((q = 0, \dot{q} = 0)\) is stable.

(b) [5 pts] with \( u = -K_d \dot{q} \), where \( K_d \) is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) [10 pts] with \( u = g(q) - K_p (q - q^*) - K_d \dot{q} \), where \( K_p \) and \( K_d \) are positive diagonal matrices and \( q^* \) is a desired robot position in \( \mathbb{R}^n \), show that the point \((q = q^*, \dot{q} = 0)\) is an asymptotically stable equilibrium point.

4. Design of a stabilizing controller for a simple mechanical system and Matlab implementation.

Consider an omnidirectional hovercraft (Fig. 1) modeled as a fully actuated rigid body in the plane. It has mass \( m \) and moment of inertia \( J \). It is controlled with three bidirectional thrusters. Two of them are placed in the rear at distance \( r \) from the central axis, and the third passes through the body laterally aligned with the center of mass. The hovercraft position is denoted by \( p = (x, y) \) and its orientation by \( \theta \). The system coordinates are \( q = (x, y, \theta) \). The forces produced by each thruster are denoted by \( u = (u_1, u_2, u_3) \).

The equations of motion of the system can be expressed as

\[
M \ddot{q} + D \dot{q} = B(q)u,
\]

where

\[
M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix}, \quad D = \begin{pmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_\theta \end{pmatrix}, \quad B(q) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -r & r & 0 \end{pmatrix},
\]

with \( d_x, d_y, d_\theta > 0 \) defining viscous damping constants.

**Analytical Problems:**

(a) [10 pts] Design an exponentially stable controller as a function of the state, i.e. \( u = k(q, \dot{q}) \), so that the system can stabilizes at the origin \((q, \dot{q}) = (0, 0)\). Prove that your controller is exponentially stable.

---

\(^1\)A skew-symmetric matrix \( S \in \mathbb{R}^{n \times n} \) has the property that \( x^T S x = 0 \) for all \( x \in \mathbb{R}^n \).
(b) [10 pts] Imagine that there is a disk-like obstacle at position \( p_o = (x_o, y_o) \) with radius \( r_o \) that the vehicle must avoid. Assume that the vehicle can sense the obstacle if it is within \( d_o \) meters of it. Augment your control law with an obstacle avoidance term which applies a “steering” force to the \((x, y)\) degrees of freedom defined by

\[
\begin{pmatrix}
  f_x \\
  f_y
\end{pmatrix} = \frac{k_o}{d(q)} \begin{bmatrix}
  0 & -1 \\
  1 & 0
\end{bmatrix} \begin{pmatrix}
  \dot{x} \\
  \dot{y}
\end{pmatrix}.
\]

The force is applied only when the vehicle is heading towards an obstacle, i.e. when the angle between the velocity \((\dot{x}, \dot{y})\) and direction towards obstacle is less than \(\pi/2\). Here, \(k_o\) is positive constant and \(d(q) = \sqrt{(x-x_o)^2 + (y-y_o)^2} - r_o\) is the distance between the vehicle and obstacle. Prove the system is globally asymptotically stable.

**Implementation:**

Choose the following model parameters: \(m = 1, J = .1, r = .2, D = \text{diag}(.01, .1, .02)\).

(a) [5 pts] Obstacle-free case: implement the controller and simulate the closed-loop system from two initial conditions. In both cases set \(q(0) = (3, 2, -\pi/4)\). The first initial condition must be with zero velocity (i.e. \(\dot{q}(0) = 0\)), while the second with non-zero velocity that you’re free to choose.

(b) [5 pts] Obstacle avoidance case: add an obstacle with \(r_0 = .25\) at position \(p_o = (1, 1)\) and set \(d_o = 1\). Design and simulate the obstacle avoidance controller from the two initial conditions specified in a). Generate trajectories for a few different choices of \(k_o\) and comment on the effect of this gain.

An example implementation of a simpler point-mass vehicle stabilization with obstacle avoidance is provided for reference. See file `hw2_example.m`.

Note: Upload your code and plots as a .zip file using [https://forms.gle/Z2AYx3FRNJHtXTR47](https://forms.gle/Z2AYx3FRNJHtXTR47) in addition attach a printout of the code and all plots to your homework solutions.

5. [Extra Credit - 5 pts] (Khalil) Consider the system

\[
\dot{x} = -a[I_n + S(x) + xx^T]x,
\]

where \(a\) is a positive constant, \(I_n\) is the nxn identity matrix, and \(S(x)\) is an \(x\)-dependent skew symmetric matrix. Note that this system is the same as in hw#1. Show that the origin is globally exponentially stable.