1 Introduction

The role of distributions and controllability:

- *distributions* determine possible directions of motion
- *nonlinear controllability* determines which states can be reached
- *motion planning* employs these structural properties to generate trajectories
- *trajectory tracking* processes feedback to follow these trajectories

*today’s slides adapted from G. Oriolo with permission*

2 Stabilizability of Nonholonomic Systems

Given a nonlinear control system

\[ \dot{x} = f(x, u) \]

our goal is to construct a control law

\[ u = k(x) \]

which accomplishes:

- *stabilization*: an equilibrium point \( x_e \) is made asymptotically stable, or
- *tracking*: a desired feasible trajectory \( x_d(t) \) is asymptotically stable

the linear approximation of the system at \( x_e \) is

\[ \delta x = A\delta x + B\delta u \quad \delta x = x - x_e, \delta u = u - u_e, \]

where \( A \triangleq \partial_x f(x_e, u_e), B \triangleq \partial_u f(x_e, u_e) \)

- if the linearized system is controllable, then the nonlinear system can be locally smoothly stabilized at \( x_e \) using a feedback law \( \delta u = K\delta x \)
- recall that the linear system is controllable if

\[ \text{rank}([B \ AB \ \cdots \ A^{n-1}B]) = n \]
• for driftless (kinematic) models $\dot{q} = G(q)u$ the linear approximation around $x_e$ has always uncontrollable eigenvalues at zero since

$$A = 0 \quad \text{and} \quad \text{rank} B = \text{rank} G(q_e) = m \leq n$$

• Necessary conditions by Brockett’s Theorem: If the system

$$\dot{x} = f(x, u)$$

is locally asymptotically $C^1$-stabilizable at $x_e = 0$ then the image of the map

$$f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$$

contains some neighborhood of $x_e$. More formally, $\exists \delta > 0$, s.t. $\forall \|\xi\| \leq \delta, \exists x, u$ such that $f(x, u) = \xi$.

• For the special case

$$\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m$$

with linearly independent control vectors $g_i$ and

$$\text{rank} \{[g_1(x_e), \ldots, g_m(x_e)]\} = m$$

the system is asymptotically $C^1$-stabilizable at $x_e$ if and only if $m \geq n$

• Therefore, nonholonomic mechanical systems cannot be stabilized at a point by smooth feedback

• The alternatives are: 1) time-varying feedback; 2) non-smooth (e.g. switching) feedback

3 Steering methods for chained forms (optional material)

3.1 Overview

• the objective is to build a sequence of open-loop input commands that steer the system from $q_i$ to $q_f$ satisfying the nonholonomic constraints

• the degree of nonholonomy gives a good measure of the complexity of the steering algorithm

• there exist canonical model structures for which the steering problem can be solved efficiently

  – chained form
  – power form
  – Chaplygin form

• interest in the transformation of the original model equation into one of these forms

• such model structures allow also a simpler design of feedback stabilizers (necessarily, non-smooth or time-varying)

• we limit the analysis to the case of systems with two inputs, where the three above forms are equivalent (via a coordinate transformation)
3.2 Chained Forms [Murray and Sastry 1993]

- a $(2,n)$ chained form is a two-input driftless control system

\[ \dot{z} = g_1(z)v_1 + g_2(z)v_2 \]

in the following form

\[
\begin{align*}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2v_1 \\
&\vdots \\
\dot{z}_n &= z_{n-1}v_1 
\end{align*}
\]

- denoting the repeated Lie brackets as $\text{ad}^k_{g_1,g_2}$

\[
\text{ad}_{g_1,g_2} = [g_1,g_2], \quad \text{ad}^k_{g_1,g_2} = [g_1,\text{ad}^k_{g_1,g_2}]
\]

one has

\[
g_1 = \begin{pmatrix}
1 \\
0 \\
z_2 \\
z_3 \\
& \ddots \\
z_{n-1}
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
& \ddots \\
0
\end{pmatrix} \Rightarrow \text{ad}^k_{g_1,g_2} = \begin{pmatrix}
0 \\
& \ddots \\
& & 0 \\
& & & (-1)^k
\end{pmatrix}
\]

in which $(-1)^k$ is the $(k+2)$-th entry.

- a one-chain system is completely nonholonomic (controllable) since the $n$ vectors

\[
\{g_1, g_2, \ldots, \text{ad}^k_{g_1,g_2}, \ldots\}, \quad i = 1, \ldots, n-2
\]

are independent

- its degree of nonholonomy is $k = n - 1$

- $v_1$ is called the generating input, $z_1$ and $z_2$ are called base variables

- if $v_1$ is (piecewise) constant, the system in chained form behaves like a (piecewise) linear system

- chained systems are a generalization of first- and second-order controllable systems for which sinusoidal steering from $z_i$ to $z_f$ minimizes the integral norm of the input

- different input commands can be used, e.g.

  - sinusoidal inputs
  - piecewise constant inputs
  - polynomial inputs
3.3 Steering with polynomial inputs

- idea similar to piecewise constant input, but with improved smoothness properties w.r.t. time
  (remember that kinematic models are controlled at the (pseudo)velocity level)

- the controls are chosen as
  \[ v_1 = \frac{z_{f1} - z_{01}}{T}, \]
  \[ v_2 = c_0 + c_1 t + \ldots + c_{n-2} t^{n-2} \]

where \( T \) is desired final time and \( c_0, \ldots, c_n \) obtained solving the linear system coming from
the closed-form integration of the model

\[
M(T) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix} + m(z_0, T) = \begin{pmatrix} z_{f2} \\ z_{f3} \\ \vdots \\ z_{fn} \end{pmatrix}
\]

with \( M(T) \) nonsingular for \( T \neq 0 \).

- if \( z_{f1} = z_{01} \) and intermediate point must be added

**Example 1.** Unicycle: consider the following change of coordinates

\[
z_1 = x \\
z_2 = \tan \theta \\
z_3 = y.
\]

and input variables

\[
u_1 = v_1 / \cos \theta \\
u_2 = v_2 \cos^2 \theta.
\]

The new equivalent system becomes

\[
\dot{z}_1 = v_1 \\
\dot{z}_2 = v_2 \\
\dot{z}_3 = z_2 v_1,
\]

Assume that the system must move between two configurations which we express in terms of the
new coordinates by \((z_{01}, z_{02}, z_{03})\) (initial) and \((z_{f1}, z_{f2}, z_{f3})\) (final).

To satisfy the first coordinate we set

\[
v_1 = \frac{z_{f1} - z_{01}}{T}, \quad v_2 = c_0 + c_1 t,
\]

where \( c_0, c_1 \) are unknowns. After integrating \( \dot{z}_2 \) we have

\[
z_2(t) = z_{02} + c_0 t + \frac{1}{2} c_1 t^2
\]
from which after integrating \( \dot{z}_3 \) we get

\[
z_3(t) = z_{03} + v_1 \left( z_{02}t + \frac{1}{2} c_0 t^2 + \frac{1}{6} c_1 t^3 \right)
\]

Now we can solve for \( c_0, c_1 \) the relationships

\[
z_2(T) = z_{f2}, \quad z_3(T) = z_{f3}
\]

which is equivalent to the relationship

\[
\begin{bmatrix}
T \\
\frac{T^2}{2} \\
\frac{T^3}{6}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1
\end{bmatrix} +
\begin{bmatrix}
z_{02} \\
z_{03} + v_1 z_{02} T
\end{bmatrix} =
\begin{bmatrix}
z_{f2} \\
z_{f3}
\end{bmatrix}
\]

and so the coefficients are found as

\[
\begin{bmatrix}
c_0 \\
c_1
\end{bmatrix} = M(T)^{-1} \left( \begin{bmatrix}
z_{f2} \\
z_{f3}
\end{bmatrix} - m(z_0, T) \right).
\]

- Sinusoidal inputs: a two-phase process
  - Phase 1: steer base variables \( z_1 \) and \( z_2 \) to their desired values \( z_{f1} \) and \( z_{f2} \)
  - Phase 2: choose
    \[
    \begin{align*}
    v_1 &= a_0 + \sin \omega t \\
    v_2 &= b_0 + \cos \omega t + \cdots + b_{n-2} \cos(n-2)\omega t,
    \end{align*}
    \]
    and solve numerically for the \( n+1 \) unknowns in terms of the boundary conditions

- piece-wise constant controls
  - subdivide total time \( T \) into subintervals of length \( \delta \)
    \[
    \begin{align*}
    v_1(\tau) &= v_{1,k} \quad \tau \in [(k-1)\delta, k\delta] \\
    v_2(\tau) &= v_{2,k}
    \end{align*}
    \]
  - it is convenient to set \( v_1 = \text{constant} \Rightarrow \) then the unknowns
    \[
    v_{2,1}, v_{2,2}, \ldots, v_{2,n-1}
    \]
    are found by solving a triangular linear system

### 3.4 Chained Form Transformation

Define the distributions

\[
\begin{align*}
\Delta_0 &= \text{span}\{g_1, g_2, \text{ad}_{g_1}g_2, \ldots, \text{ad}_{g_1}^{n-2}g_2\} \\
\Delta_1 &= \text{span}\{g_2, \text{ad}_{g_1}g_2, \ldots, \text{ad}_{g_1}^{n-2}g_2\} \\
\Delta_2 &= \text{span}\{g_2, \text{ad}_{g_1}g_2, \ldots, \text{ad}_{g_1}^{n-3}g_2\}
\end{align*}
\]
If, for some open set, one has (i) \( \dim \Delta_0 = n \), (ii) \( \Delta_1, \Delta_2 \) are involutive, (iii) there exists a function \( h_1 \) such that
\[
dh_1 \cdot \Delta_1 = 0 \quad dh_1 \cdot g_1 = 1
\]
then the system can be transformed into chained form

the change of coordinates is given by

\[
\begin{align*}
z_1 &= h_1 \\
z_2 &= L_{g_1}^{n-2}h_2 \\
    & \quad \vdots \\
z_{n-1} &= L_{g_1}h_2 \\
z_n &= h_2
\end{align*}
\]

with \( h_2 \) independent from \( h_1 \) and such that \( dh_2 \cdot \Delta_2 = 0 \) the input transformation is given by

\[
\begin{align*}
v_1 &= u_1 \\
v_2 &= (L^{n-1}_{g_1}h_2)u_1 + (L_{g_2}L_{g_1}^{n-2}h_2)u_2
\end{align*}
\]