1 Distributions

The role of distributions and controllability:

- **distributions** determine possible directions of motion
- **nonlinear controllability** determines which states can be reached
- **motion planning** employs these structural properties to generate trajectories
- **trajectory tracking** processes feedback to follow these trajectories

- Let \( g_1(x), \ldots, g_m(x) \) be linearly independent vector fields on \( M \).
- A **distribution** \( \Delta \) assigns a subspace of the tangent space to each point defined by
  \[
  \Delta = \text{span}\{g_1, \ldots, g_m\}.
  \]

- A distribution \( \Delta \) is **involutive** if it is closed under the Lie bracket, i.e. if
  \[
  \forall f(x), g(x) \in \Delta(x), \quad [f(x), g(x)] \in \Delta(x)
  \]

- A distribution \( \Delta \) is **regular** if the dimension of \( \Delta_x \) does not vary with \( x \).

- A distribution \( \Delta \) of constant dimension \( k \) is **integrable** if for every \( x \in \mathbb{R}^n \) there are smooth functions \( h_i : \mathbb{R}^n \to \mathbb{R} \) such that \( \frac{\partial h_i}{\partial x} \) are linearly independent at \( x \) and for every \( f \in \Delta \)
  \[
  L_fh_i = \frac{\partial h_i}{\partial x}f(x) = 0, \quad i = 1, \ldots, n-k.
  \]

- The hypersurfaces defined as the level sets
  \[
  \{q : h_1(x) = c_1, \ldots, h_{n-k}(x) = c_{n-k}\},
  \]
  are called **integral manifolds** for the distribution.

- **Frobenius Theorem:** A regular distribution is integrable if and only if is involutive.

- If the distribution \( \Delta \) is involutive then its integral manifolds (level sets of functions \( h_i \)) are **leaves** of a **foliation** of \( \mathbb{R}^n \)
Examples

- The nonholonomic integrator

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
-x_2
\end{pmatrix} u_1 + \begin{pmatrix}
0 \\
1 \\
x_1
\end{pmatrix} u_2
\]

- Trapped on a sphere

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
x_2 \\
-x_1 \\
0
\end{pmatrix} u_1 + \begin{pmatrix}
x_3 \\
0 \\
-x_1
\end{pmatrix} u_2
\]

2 Nonlinear Controllability

2.1 Reachable Sets

- Consider the nonlinear control system (NCS)

\[
\Sigma : \quad \dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x \in \mathbb{R}^n, u \in U \subset \mathbb{R}^m
\]

- A system is controllable if for any \( x_0, x_f \in \mathbb{R}^n \) there exists a time \( T \) and \( u : [0, T] \rightarrow U \) such that \( \Sigma \) satisfies \( x(0) = x_0 \) and \( x(T) = x_f \).

- A system is small-time locally controllable (STLC) at \( x_0 \) if it can reach nearby points in arbitrary small times and stay near \( x_0 \).

- The reachable set \( \mathcal{R}^V(x_0, T) \) is the set of states \( x(T) \) for which there is a control \( u : [0, T] \rightarrow U \) that steers the system from \( x(0) \) to \( x(T) \) without leaving an open set \( V \) around \( x_0 \).

- The set of states reachable up to time \( T \) is defined by

\[
\mathcal{R}^V(x_0, \leq T) = \bigcup_{0<\tau\leq T} \mathcal{R}^V(x_0, \tau)
\]

2.2 Controllability Conditions

- NCS is locally accessible (LA) from \( x_0 \) if \( \forall V, \) a neighborhood of \( x_0 \) and \( \forall T > 0 \)

\[
\Omega \subset \mathcal{R}^V(x_0, \leq T), \text{ for some open set } \Omega
\]

- NCS is STLC if every neighborhood \( V \) of \( x_0 \) and every \( T > 0 \) if \( \mathcal{R}^V(x_0, T) \) contains a neighborhood of \( x_0 \).

- STLC \( \Rightarrow \) controllability \( \Rightarrow \) LA (not vice versa)
• Checking LA is performed through an algebraic test:
  – Let $\bar{\Omega}$ be the involutive closure of the distribution of $\{g_0, g_1, \ldots, g_m\}$
  – **Theorem (Chow):** NCS is LA from $x_0$ if and only if
    \[
    \dim \bar{\Delta}(x_0) = n : \text{accessibility rank condition}
    \]
  – Algorithmic Test:
    \[
    \bar{\Delta} = \text{span} \left\{ v \in \bigcup_{k \geq 0} \Delta^k \right\} \text{ with } \begin{cases}
        \Delta^0 = \text{span}\{g_0, g_1, \ldots, g_m\} \\
        \Delta^k = \Delta^{k-1} + \text{span}\{[g_j, v], j = 0, \ldots, m : v \in \Delta^{k-1}\}
    \end{cases}
    \]

• only sufficient conditions exists for STLC, e.g., [Sussmann 1987]
• however, for driftless control systems:
  \[\text{LA } \iff \text{controllability } \iff \text{STLC}\]
• this equivalence holds also whenever
  \[
g_0(x) \in \text{span}\{g_1(x), \ldots, g_m(x)\}, \quad \forall x \in X
  \]
  ("trivial" drift)
• if the driftless control system
  \[
  \dot{q} = \sum_{i=1}^{m} g_i(q)v_i,
  \]
  with state $q$ and inputs $v$ is controllable, then its *dynamic extension*
  \[
  \dot{q} = \sum_{i=1}^{m} g_i(x)v_i, \\
  \dot{v}_i = u_i, \quad i = 1, \ldots, m,
  \]
  with state $x = (q, v)$ and controls $u$ is also controllable (and vice versa).

Examples
• The unicycle
  \[
  g_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad g_3 = [g_1, g_2] = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}
  \]
  \[
  \dim \bar{\Delta} = 3 \text{ for all } q
  \]
• The car-like robot (rear-drive)

\[
g_1 = \begin{pmatrix}
\cos \theta \\
\sin \theta \\
\tan \phi / \ell
\end{pmatrix},
g_2 = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

\[
g_3 = [g_1, g_2] = \begin{pmatrix}
0 \\
0 \\
-1 / \ell \cos^2 \phi \\
0
\end{pmatrix},
g_4 = [g_1, g_3] = \begin{pmatrix}
-\sin \theta / \ell \cos^2 \phi \\
\cos \theta / \ell \cos^2 \phi \\
0 \\
0
\end{pmatrix}
\]

\[\dim \Delta = 4\] away from singularity at \( \phi = \pm \pi/2 \) of \( g_1 \)

• more generally, the filtration of a distribution \( \Delta \) is defined by

\[
\Delta_1 = \Delta, \quad \Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}], i \geq 2
\]

where

\[ [\Delta_1, \Delta_{i-1}] = \text{span}\{[g, h] : g \in \Delta_1, h \in \Delta_{i-1}\} \]

• after enough “bracketing” (e.g. \( k \) times) the rank of \( \Delta_i \) for \( i \geq k \) stops increasing, no more new directions of motion appear. The smallest such \( k \) is called degree of nonholonomy of the distribution, i.e. such that

\[ \dim \Delta_{k+1} = \dim \Delta_k. \]

• Classification in terms of \( k \)
  
  – completely nonholonomic: \( \dim(\Delta_k) = n \)
  
  – partially nonholonomic: \( m < \dim(\Delta_k) < n \)
  
  – holonomic: \( \dim(\Delta_k) = m = n - k \)

• Examples: unicycle \((k = 2)\), car-like robot \((k = 3)\)

2.3 Good and bad brackets

For the general system with non-zero drift \( g_0 \) term we will use the concept of good and bad brackets.

A bad bracket is a Lie bracket generated using an odd number of \( g_0 \) vectors and even number of \( g_i \) (for each \( i = 1, \ldots, m \)) vectors. A good bracket is one that is not bad.

**Theorem 1.** A control system with \( x \in \mathbb{R}^n \) and controls \( u \in U \subset \mathbb{R}^m \)

\[
\dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i
\]

is STLC at \( x^* \) if

1. \( g_0(x^*) = 0 \)
2. \( U \) is open and its convex hull contains 0
3. LARC is satisfied using brackets of degree $k$

4. any bad bracket of degree $j \leq k$ can be expressed as linear combination of good brackets of degree $< j$

**Example 1.** from Principles of Robot Motion Consider the planar rigid body with state $x \in \mathbb{R}^6$ defining its position, orientation, and velocities, controls $u \in \mathbb{R}^2$ defining the forward force and lateral force (at distance $d$ from the center-of-mass) with vector fields

$$g_0(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin x_3 \\ \cos x_3 \\ -d \end{pmatrix},$$

We can define

$$g_3 = [g_0, g_1], \quad g_4 = [g_0, g_2], \quad g_5 = [g_1, g_4], \quad g_6 = [g_0, g_5],$$

and note that

$$\det([g_1, g_2, g_3, g_4, g_5, g_6]) = d^4 \Rightarrow \text{LARC of degree 4}$$

The bad brackets of degree $< 4$ are

$$[g_1, [g_0, g_1]] = 0, \quad [g_2, [g_0, g_2]] = (0, 0, 0, 2d \cos x_3, 2d \sin x_3, 0) \triangleq 2dg_1,$$

and since both are spanned by good brackets of lower order then the system is STLC. Note that since the first bad bracket is zero then it becomes irrelevant.