1 Introduction
The role of distributions and controllability:

- **distributions** determine possible directions of motion
- **nonlinear controllability** determines which states can be reached
- **motion planning** employs these structural properties to generate trajectories
- **trajectory tracking** processes feedback to follow these trajectories

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2 Stabilizability of Nonholonomic Systems
Given a nonlinear control system
\[ \dot{x} = f(x, u) \]
our goal is to construct a control law
\[ u = k(x) \]
which accomplishes:

- **stabilization**: an equilibrium point \( x_e \) is made asymptotically stable, or
- **tracking**: a desired feasible trajectory \( x_d(t) \) is asymptotically stable
- the linear approximation of the system at \( x_e \) is
\[ \delta x = A\delta x + B\delta u \quad \delta x = x - x_e, \delta u = u - u_e, \]
where \( A \triangleq \partial_x f(x_e, u_e), B \triangleq \partial_u f(x_e, u_e) \)
- if the linearized system is controllable, then the nonlinear system can be locally smoothly stabilized at \( x_e \) using a feedback law \( \delta u = K\delta x \)
- recall that the linear system is controllable if
\[ \text{rank}([B \ AB \ \cdots \ A^{n-1}B]) = n \]
• for driftless (kinematic) models \( \dot{q} = G(q)u \) the linear approximation around \( x_e \) has always uncontrollable eigenvalues at zero since
\[
A = 0 \quad \text{and} \quad \text{rank}B = \text{rank}(G(q_e)) = m \leq n
\]
• Necessary conditions by Brockett’s Theorem: If the system
\[
\dot{x} = f(x, u)
\]
is locally asymptotically \( C^1 \)-stabilizable at \( x_e = 0 \) then the image of the map
\[
f : \mathbb{R}^n \times U \to \mathbb{R}^n
\]
contains some neighborhood of \( x_e \). More formally, \( \exists \delta > 0 \), s.t. \( \forall \|\xi\| \leq \delta, \exists x, u \) such that \( f(x, u) = \xi \).
• For the special case
\[
\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m
\]
with linearly independent control vectors \( g_i \) and
\[
\text{rank}\{[g_1(x_e), \ldots, g_m(x_e)]\} = m
\]
the system is asymptotically \( C^1 \)-stabilizable at \( x_e \) if and only if \( m \geq n \)
• Therefore, nonholonomic mechanical systems cannot be stabilized at a point by smooth feedback.
• The alternatives are: 1) time-varying feedback; 2) non-smooth (e.g. switching) feedback.

3 Steering methods for chained forms (optional material)

3.1 Overview
• the objective is to build a sequence of open-loop input commands that steer the system from \( q_i \) to \( q_f \) satisfying the nonholonomic constraints.
• the degree of nonholonomy gives a good measure of the complexity of the steering algorithm.
• there exist canonical model structures for which the steering problem can be solved efficiently.
  – chained form
  – power form
  – Chaplygin form
• interest in the transformation of the original model equation into one of these forms.
• such model structures allow also a simpler design of feedback stabilizers (necessarily, non-smooth or time-varying).
• we limit the analysis to the case of systems with two inputs, where the three above forms are equivalent (via a coordinate transformation).
3.2 Chained Forms [Murray and Sastry 1993]

- A (2, n) chained form is a two-input driftless control system

\[ \dot{z} = g_1(z)v_1 + g_2(z)v_2 \]

in the following form

\[ \begin{align*}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2v_1 \\
& \vdots \\
\dot{z}_n &= z_{n-1}v_1
\end{align*} \]

- Denoting the repeated Lie brackets as \( \text{ad}_{g_1}^k g_2 \)

\[ \text{ad}_{g_1} g_2 = [g_1, g_2], \quad \text{ad}_{g_1}^k g_2 = [g_1, \text{ad}_{g_1}^{k-1} g_2] \]

one has

\[ g_1 = \begin{pmatrix}
1 \\
0 \\
z_2 \\
z_3 \\
& \vdots \\
z_{n-1}
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
& \vdots \\
0
\end{pmatrix} \Rightarrow \text{ad}_{g_1}^k g_2 = \begin{pmatrix}
0 \\
& \vdots \\
0
\end{pmatrix} \]

in which \((-1)^k\) is the \((k+2)\)-th entry.

- A one-chain system is completely nonholonomic (controllable) since the \(n\) vectors

\[ \{g_1, g_2, \ldots, \text{ad}_{g_1}^i g_2, \ldots\}, \quad i = 1, \ldots, n - 2 \]

are independent

- Its degree of nonholonomy is \(k = n - 1\)

- \(v_1\) is called the generating input, \(z_1\) and \(z_2\) are called base variables

- If \(v_1\) is (piecewise) constant, the system in chained form behaves like a (piecewise) linear system

- Chained systems are a generalization of first- and second-order controllable systems for which sinusoidal steering from \(z_i\) to \(z_f\) minimizes the integral norm of the input

- Different input commands can be used, e.g.

  - Sinusoidal inputs
  - Piecewise constant inputs
  - Polynomial inputs
### 3.3 Steering with polynomial inputs

- Idea similar to piecewise constant input, but with improved smoothness properties w.r.t. time (remember that kinematic models are controlled at the (pseudo)velocity level).
- The controls are chosen as:
  
  \[ v_1 = \frac{z_{f1} - z_{01}}{T}, \]
  \[ v_2 = c_0 + c_1 t + \ldots + c_{n-2} t^{n-2} \]

  where \( T \) is desired final time and \( c_0, \ldots, c_n \) obtained solving the linear system coming from the closed-form integration of the model

  \[
  M(T) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix} + m(z_0, T) = \begin{pmatrix} z_{f2} \\ z_{f3} \\ \vdots \\ z_{fn} \end{pmatrix}
  \]

  with \( M(T) \) nonsingular for \( T \neq 0 \).

- If \( z_{f1} = z_{01} \) and intermediate point must be added.

**Example 1.** Unicycle: consider the following change of coordinates

\[
\begin{align*}
z_1 &= x \\
z_2 &= \tan \theta \\
z_3 &= y.
\end{align*}
\]

and input variables

\[
\begin{align*}
u_1 &= v_1 \cos \theta \\
u_2 &= v_2 \cos^2 \theta.
\end{align*}
\]

The new equivalent system becomes

\[
\begin{align*}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1,
\end{align*}
\]

Assume that the system must move between two configurations which we express in terms of the new coordinates by \((z_{01}, z_{02}, z_{03})\) (initial) and \((z_{f1}, z_{f2}, z_{f3})\) (final).

To satisfy the first coordinate we set

\[
\begin{align*}
v_1 &= \frac{z_{f1} - z_{01}}{T}, \\
v_2 &= c_0 + c_1 t,
\end{align*}
\]

where \(c_0, c_1\) are unknowns. After integrating \(\dot{z}_2\) we have

\[
z_2(t) = z_02 + c_0 t + \frac{1}{2} c_1 t^2
\]
from which after integrating \( \dot{z}_3 \) we get

\[
z_3(t) = z_{03} + v_1 \left( z_{02}t + \frac{1}{2}c_0 t^2 + \frac{1}{6}c_1 t^3 \right)
\]

Now we can solve for \( c_0, c_1 \) the relationships

\[
z_2(T) = z_{f2}, \quad z_3(T) = z_{f3}
\]

which is equivalent to the relationship

\[
\begin{bmatrix}
    T \\
    \frac{T^2}{2} \\
    \frac{T^3}{6}
\end{bmatrix}
\begin{bmatrix}
    c_0 \\
    c_1
\end{bmatrix}
+ \begin{bmatrix}
    z_{02} \\
    z_{03} + v_1 z_{02} T
\end{bmatrix}
= \begin{bmatrix}
    z_{f2} \\
    z_{f3}
\end{bmatrix}
\]

and so the coefficients are found as

\[
\begin{bmatrix}
    c_0 \\
    c_1
\end{bmatrix}
= (M(T))^{-1} \left( \begin{bmatrix}
    z_{f2} \\
    z_{f3}
\end{bmatrix}
- m(z_0, T) \right).
\]

• Sinusoidal inputs: a two-phase process
  - Phase 1: steer base variables \( z_1 \) and \( z_2 \) to their desired values \( z_{f1} \) and \( z_{f2} \)
  - Phase 2: choose

\[
\begin{align*}
v_1 &= a_0 + \sin \omega t \\
v_2 &= b_0 + \cos \omega t + \ldots + b_{n-2} \cos(n-2)\omega t,
\end{align*}
\]

and solve numerically for the \( n+1 \) unknowns in terms of the boundary conditions

• piece-wise constant controls
  - subdivide total time \( T \) into subintervals of length \( \delta \)

\[
\begin{align*}
v_1(\tau) &= v_{1,k} \\
v_2(\tau) &= v_{2,k}, \quad \tau \in [(k-1)\delta, k\delta]
\end{align*}
\]

  - it is convenient to set \( v_1 = \text{constant} \) \( \Rightarrow \) then the unknowns

\[
v_{2,1}, v_{2,2}, \ldots, v_{2,n-1}
\]

are found by solving a triangular linear system

### 3.4 Chained Form Transformation

Define the distributions

\[
\begin{align*}
\Delta_0 &= \text{span}\{g_1, g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-2} g_2\} \\
\Delta_1 &= \text{span}\{g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-2} g_2\} \\
\Delta_2 &= \text{span}\{g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-3} g_2\}
\end{align*}
\]
If, for some open set, one has (i) \( \dim \Delta_0 = n \), (ii) \( \Delta_1, \Delta_2 \) are involutive, (iii) there exists a function \( h_1 \) such that
\[
dh_1 \cdot \Delta_1 = 0 \quad dh_1 \cdot g_1 = 1
\]
then the system can be transformed into chained form.

the change of coordinates is given by
\[
\begin{align*}
z_1 &= h_1 \\
z_2 &= L_{g_1}^{n-2} h_2 \\
    &\vdots \\
z_{n-1} &= L_{g_1} h_2 \\
z_n &= h_2
\end{align*}
\]
with \( h_2 \) independent from \( h_1 \) and such that \( dh_2 \cdot \Delta_2 = 0 \) the input transformation is given by
\[
\begin{align*}
v_1 &= u_1 \\
v_2 &= (L_{g_1}^{n-1} h_2) u_1 + (L_{g_2} L_{g_1}^{n-2} h_2) u_2
\end{align*}
\]