EN530.678 Nonlinear Control and Planning in Robotics
Lecture 5: Distributions and Controllability
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1 Distributions

`These notes are still under construction.` ToDo

The role of distributions and controllability:

- *distributions* determine possible directions of motion
- *nonlinear controllability* determines which states can be reached
- *motion planning* employs these structural properties to generate trajectories
- *trajectory tracking* processes feedback to follow these trajectories
- Let $g_1(x), ..., g_m(x)$ be linearly independent vector fields on M.
- A *distribution* Δ assigns a subspace of the tangent space to each point defined by

$$\Delta = \text{span}\{g_1, ..., g_m\}.$$

- A distribution Δ is *involutive* it is closed under the Lie bracket, i.e. if

$$\forall f(x), g(x) \in \Delta(x), \quad [f(x), g(x)] \in \Delta(x)$$

- A distribution Δ is *regular* if the dimension of Δ_x does not vary with x.
- A distribution Δ of constant dimension k is *integrable* if for every $x \in \mathbb{R}^n$ there are smooth functions $h_i : \mathbb{R}^n \to \mathbb{R}$ such that $\frac{\partial h_i}{\partial x}$ are linearly independent at x and for every $f \in \Delta$

$$L_f h_i = \frac{\partial h_i}{\partial q} f(x) = 0, i = 1, ..., n-k.$$

- The hypersufraces defined as the level sets

$$\{q : h_1(x) = c_1, ..., h_{n-k}(x) = c_{n-k}\},$$

are called *integral manifolds* for the distribution.

- **Frobenius Theorem:** *A regular distribution is integrable if and only if is involutive.*

- If the distribution Δ is involutive then its integral manifolds (level sets of functions h_i) are *leaves* of a *foliation* of $\mathbb{R}^n$

Examples

- The nonholonomic integrator

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix} u_2$$

- Trapped on a sphere

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} u_2$$

2 Nonlinear Controllability

Reachable Sets

- Consider the nonlinear control system (NCS)

$$\Sigma : \qquad \dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x \in \mathbb{R}^n, u \in U \subset \mathbb{R}^m$$

- A system is *controllable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a time T and $u : [0,T] \to U$ such that Σ satisfies $x(0) = x_0$ and $x(T) = x_f$.

- A system is *small-time locally controllable* (STLC) at x_0 if it can reach nearby points in arbitrary small times and stay near x_0.

- The *reachable set* $\mathcal{R}^V(x_0, T)$ is the set of states $x(T)$ for which there is a control $u : [0,T] \to U$ that steers the system from $x(0)$ to $x(T)$ without leaving an open set V around x_0.

- The set of states reachable *up to* time T is defined by

$$\mathcal{R}^V(x_0, \leq T) = \bigcup_{0<\tau\leq T} \mathcal{R}^V(x_0, \tau)$$

Controllability Conditions

- NCS is *locally accessible* (LA) from x_0 if $\forall V$, a neighborhood of x_0 and $\forall T > 0$

$$\Omega \subset \mathcal{R}^V(x_0, \leq T), \text{ for some open set } \Omega$$

- NCS is STLC if every neighborhood V of x_0 and every $T > 0$ if $\mathcal{R}^V(x_0, T)$ contains a neighborhood of x_0.
- STLC $\Rightarrow$ controllability $\Rightarrow$ LA (not vice versa)
- Checking LA is performed through an *algebraic test*:
  - Let $\bar{\Omega}$ be the involutive closure of the distribution of $\{g_0, g_1, ..., g_m\}$
  - **Theorem (Chow):**. NCS is LA from x_0 if and only if

$$\dim\overline{\Delta}(x_0) = n : \quad \textit{accessibility rank condition}$$

  - Algorithmic Test:

$$\bar{\Delta} = \text{span}\left\{ v \in \bigcup_{k \geq 0} \Delta^k \right\} \text{ with } \left\{ \begin{array}{l} \Delta^0 = \text{span}\{g_0, g_1, ..., g_m\} \\ \Delta^k = \Delta^{k-1} + \text{span}\{[g_j, v], j = 0, ..., m : v \in \Delta^{k-1}\} \end{array} \right.$$

- only sufficient conditions exists for STLC , e.g., [Sussmann 1987]
- however, for driftless control systems:

$$\text{LA} \Leftrightarrow \text{controllability} \Leftrightarrow \text{STLC}$$

- this equivalence holds also whenever

$$g_0(x) \in \text{span}\{g_1(x), ..., g_m(x)\}, \qquad \forall x \in X$$

  ("trivial" drift)
- if the driftless control system

$$\dot{q} = \sum_{i=1}^{m} g_i(q) v_i,$$

  with state q and inputs v is controllable, then its *dynamic extension*

$$\begin{aligned} \dot{q} &= \sum_{i=1}^{m} g_i(x) v_i, \\ \dot{v}_i &= u_i, \quad i = 1, ..., m, \end{aligned}$$

  with state $x = (q, v)$ and controls u is also controllable (and vice versa).

Examples

- The unicycle

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow g_3 = [g_1, g_2] = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

  $\dim\bar{\Delta} = 3$ for all q

- The car-like robot (rear-drive)

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ -1/\ell\cos^2\phi \\ 0 \end{pmatrix}, \ g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell\cos^2\phi \\ \cos\theta/\ell\cos^2\phi \\ 0 \\ 0 \end{pmatrix}$$

$\dim\bar{\Delta} = 4$ away from singularity at $\phi = \pm\pi/2$ of g_1

- more generally, the *filtration* of a distribution Δ is defind by

$$\Delta_1 = \Delta, \qquad \Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}], i \geq 2$$

where

$$[\Delta_1, \Delta_{i-1}] = \text{span}\{[g, h] : g \in \Delta_1, h \in \Delta_{i-1}\}$$

- after enough "bracketing" (e.g. k times) the rank of Δ_i for $i \geq k$ stops increasing, no more new directions of motion appear. The smallest such k is called *degree of nonholonomy* of the distribtion, i.e. such that

$$\dim\Delta_{k+1} = \dim\Delta_k.$$

- Classification in terms of k
  - completely nonholonomic: $\dim(\Delta_k) = n$
  - partially nonholonomic: $m < \dim(\Delta_k) < n$
  - holonomic: $\dim(\Delta_k) = m = n - k$
- Examples: unicycle ($k = 2$), car-like robot ($k = 3$)