EN530.678 Nonlinear Control and Planning in Robotics
Lecture 4: Manifolds and Vector Fields

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`These notes are still under construction.` ToDo

1 Manifolds

In many practical applications the inherent nature of the configuration space is different than $\mathbb{R}^n$. For instance:

- a manipulator with rotational joints lives on a torus $T^m = S^1 \times ... \times S^1$
- rigid body configuration space of frames SE(3)
- end-effector might be constrained to a sphere S^2
- Physical constraints
  - *position constraints*: contact (e.g. end-effector constrained to a surface), mechanical joints and other kinematic coupling relationships.
  - *rolling and sliding* (car cannot move sideways, knife-edge can slide forward). These are velocity constraints that result in *integral manifolds*, i.e. from integrating flow along the subspace of allowable velocities.
  - *non-smooth constraints, hybrid systems*: end-effector move freely, then lands on a rigid surface and slides; legged robot stance on a surface, leaving/landing on the surface.
- *sensing constraints*, e.g. maintain line-of-sight to a point
- in computer science: compression of high-dimensional data, i.e. to identify and model *lower-dimensional structure*. But it is used very loosely.
- mathematical physics, general relativity, etc...

A *manifold* is a set M that locally "looks like" linear space, e.g. $\mathbb{R}^n$. A *chart* on M is a subset U of M together with a bijective map $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. We usually denote $\varphi(m)$ by $(x^1, \dots, x^n)$ and call the x^i the coordinates of the point $m \in U \subset M$. Two charts U, φ and U', φ' such that $U \cap U' \neq$ are called *compatible* if $\varphi(U \cap U')$ and $\varphi(U' \cap U)$ are open subsets of $\mathbb{R}^n$ and the maps

$$\varphi' \circ \varphi^{-1}|_{\varphi(U \cap U')} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

and

$$\varphi \circ (\varphi')^{-1}|_{\varphi'(U\cap U')} : \varphi'(U \cap U') \to \varphi(U \cap U')$$

are C^∞ (smooth). Here $\varphi \circ (\varphi')^{-1}|_{\varphi'(U\cap U')}$ denotes the restriction of the map $\varphi \circ (\varphi')^{-1}$ to the set $\varphi'(U \cap U')$.

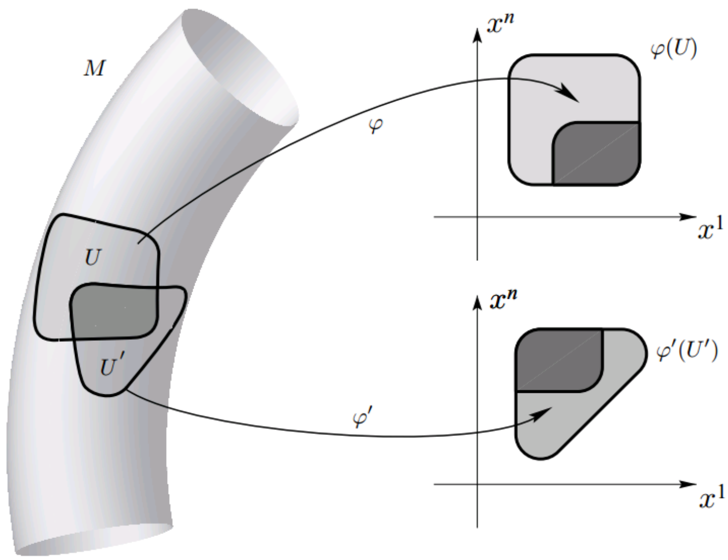

Figure 1: A manifold must be covered my overlapping charts and smooth transitions between them.

We call M a *differentiable n-manifold* when:

1. The set M is covered by a collection of charts, that is, every point is represented in at least one chart

2. M has an *atlas*; that is, M can be written as a union of compatible charts

For example, consider $\mathbb{R}^3$ as a manifold. First, we pick standard cartesian coordinates (i.e. the chart is the identity), but then add other charts such as spherical coordinates – then the collection becomes a differentiable structure (i.e. one can pass from one chart to the other smoothly). This will be understood to have been done when we say we have a manifold.

Example 1. *The circle* S^1 with charts e.g. $\varphi : S^1\backslash(-1, 0) \to (-\pi, \pi)$, and $\psi : S^1\backslash(0, -1) \to (-\pi/2, \frac{3}{2}\pi)$:

$$\varphi\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \text{atan2}(y, x), \quad \varphi^{-1}(\theta) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right],$$

where $\text{atan2}(y, x) \in (-\pi, \pi]$ and is undefined for $x = y = 0$. The second chart is basically the same, except that the lower left corner maps to angles $(\pi, \frac{3}{2}\pi)$ instead of $(-\pi, -\pi/2)$:

$$\psi\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left\{\begin{array}{ll} \theta, & \text{if } \theta \in [-\pi/2, \pi] \\ \theta + 2\pi, & \text{if } \theta \in (-\pi, -\pi/2) \end{array}\right., \quad \theta = \text{atan2}(y, x), \quad \psi^{-1} = \varphi^{-1}.$$

Example 2. S^2 *with spherical coordinates*

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \arccos(z) \\ \text{atan2}(y,x) \end{bmatrix}, \quad \varphi^{-1}\left(\begin{bmatrix} \theta \\ \varphi \end{bmatrix}\right) = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$$

Note we have $U = M \backslash \{(0,0,\pm 1)\}$ since the there is a singularity at $x = y = 0.$, which means that we need at least two charts. A second chart can be constructed by "rotating" U e.g. by $\pi/2$ degrees similarly to the S^1 example.

Definition 1. *Tangent Vectors. Two curves* $t \to c_1(t)$ *and* $t \to c_2(t)$ *in an* n*-manifold* M *are called equivalent at the point* m *if*

$$c_1(0) = c_2(0) = m,$$

and

$$\frac{d}{dt}(\varphi \circ c_1)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)\Big|_{t=0}$$

in some chart φ*.*

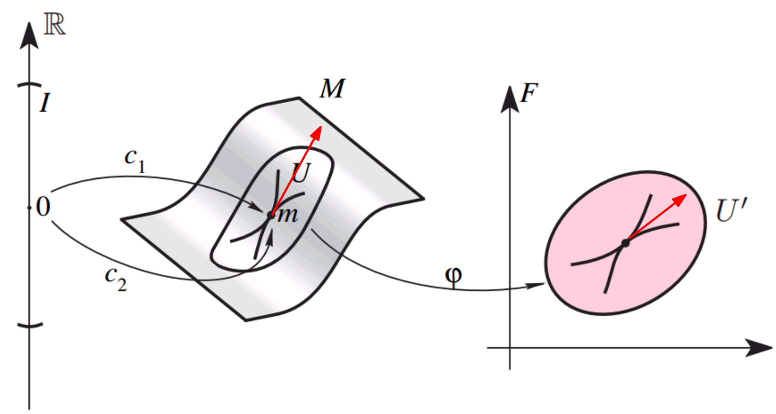

Figure 2: Consider two curves c_1 and c_2 passing through a point m on the manifold M with the same velocity vector at that point. Such velocity vectors are called tangent vectors.

A *tangent vector* v to a manifold M at point m is an equivalent class of curves at m. The set of tangent vectors to M at m is a vector space. We denoted it by $T_m M$ = *tangent space* to M at $m \in M$. We think of $v \in T_m M$ as tangent to a curve in M.

The *components* of a tangent v are the numbers $v^1, \ldots, v^n$ defined by taking derivatives of the components of the curve $\varphi \circ c$:

$$v^i = \frac{d}{dt}(\varphi \circ c)^i\Big|_{t=0}$$

The components are independent of the representative curve chosen, but they do depend on the chart chosen. (Think of *components* as the coordinates of the velocity).

Definition 2. *The tangent bundle* of M denoted by TM is the disjoint union of the tangent spaces to M at the points $m \in M$, i.e.

$$TM = \cup_{m \in M} T_m M$$

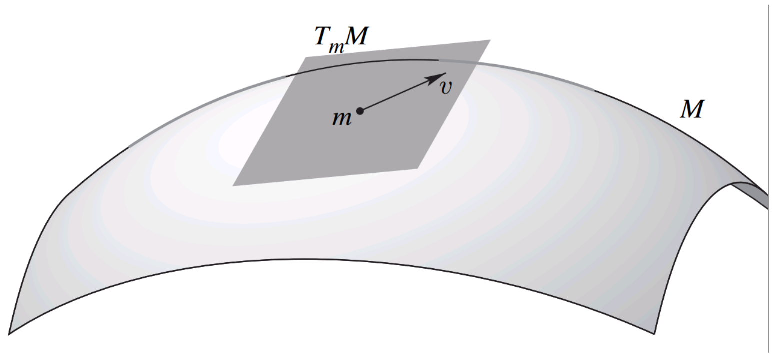

Figure 3: All tangent vectors at a point m form a vector space (e.g. they could span $\mathbb{R}^n$), called the tangent space at m.

Points in TM are vectors v tangent at some $m \in M$. If M is an n-manifold then TM is a $2n$-manifold. The *natural projection* is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $m \in M$ at which the vector v is attached. The inverse image $\tau_M^{-1}(m)$ of $m \in M$ is the tangent space T_mM – the *fiber* of TM over the point $m \in M$.

1.0.1 Vector fields.

Definition 3. *A* vector field X *on* M *is a map* $X : M \to TM$ *that assignes a vector* $X(m)$ *at the point* $m \in M$*, i.e.* $\tau_M \circ X = identity$*. The vector space of vector fields is denoted* $\mathfrak{X}(M)$.

An *integral curve* of X with initial condition m_0 at $t = 0$ is a map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = m_0$ and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$, i.e. a *solution curve* of this ODE.

The *flow* of X: a collection of maps $\Phi_t : M \to M$ such that $t \to \Phi_t(m)$ is the integral curve of X with initial condition m.

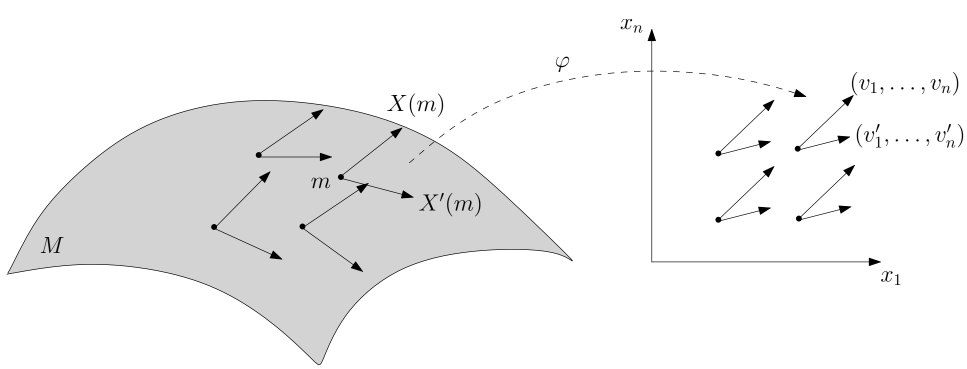

Figure 4: Two vector fields X and X' represented by local coordinates $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$.

The deriviative of $f : M \to \mathbb{R}$ at $m \in M$ gives a map $T_m f : T_m M \to T_{f(m)}\mathbb{R} \simeq \mathbb{R}$. It is actually a linear map $df(m) : T_m M \to \mathbb{R}$. Thus $df(m) \in T_m^* M$, the dual of $T_m M$ (the dual is the space of linear functions). If we replace each vector space $T_m M$ with its dual $T_m^* M$ we obtain a $2n$-manifold called the cotangent bundle and denoted by $T^* M$. We call df the *differential* of f. For every $v \in T_m M$ we call $df(m) \cdot v$ the *directional derivative* of f in the direction of v.

In a coordinate chart or in a vector space, this notion conincides with the usual notion of a directional derviative learned in vector calculus. Using a chart φ the directional derivative is

$$df(m) \cdot v = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} v^i$$

Note that with this definition we can regard vectors v as *differential operators*, i.e. which differentiate functions. In particular we can identify a basis of $T_m M$ using the operators $\frac{\partial}{\partial x^i}$ and we write

$$\{e_1, \ldots, e_n\} = \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$$

for this basis, so that $v = v^i \frac{\partial}{\partial x^i}$. In other words, think of $\frac{\partial}{\partial x^1}$ as a unit column vector $(1, 0, \ldots, 0)$ along which we can differentiate.

Tangent vectors in one chart transform to tangent vectors in another chart through the Jacobian of the map between the two charts.

Example 3. Vector field on a sphere using spherical coordinates (θ, ϕ). *An example of a vector field in a basis* $(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$ *is e.g.* $X = \theta^2 \frac{\partial}{\partial \theta} - \theta\phi \frac{\partial}{\partial \phi}$. *It can be visualized using the standard basis in* $\mathbb{R}^3$ *defined by*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \varphi^{-1}\left(\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}\right), \qquad D\varphi^{-1} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix}$$

which determines a local tangent space at each $q \in S^2$. *So the vector field* X *expressed in cartesian coordinates will look like the vector*

$$\begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \theta^2 \\ -\theta\phi \end{bmatrix}.$$

Note: at $\theta = (0, \pi)$ *the tangent space is undefined* $\Rightarrow$ *need two charts.*

There is a one to one correspondence between vector fields X on M and the differential operators

$$X[f](m) = df(m) \cdot X(m)$$

The *dual basis* to $\frac{\partial}{\partial x^i}$ is denoted by dx^i (think row unit vector so that $dx^j \frac{\partial}{\partial x^i} = 1$ only when $i = j$ and 0 otherwise). Thus, relative to a choice of local coordiantes we get the basis formula

$$df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

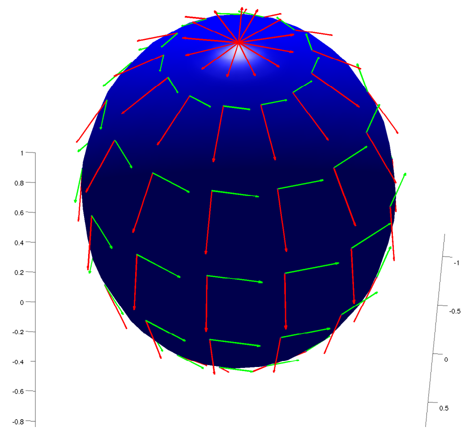

Figure 5: Tangent vectors on the sphere corresponding to basis vectors $(1,0)$ and $(0,1)$ in spherical coordinates at each piont $(x^1, x^2) = (\theta, \phi)$. These tangent vectors were computed using the jacobian of the inverse of the coordinate map φ. Note that one of the basis vectors shrinks to zero at the poles, suggesting that one chart is not enough to cover the sphere.

We also have

$$X[f] = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}$$

which is why we write

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

The *Lie derivative* $L_X f$ is another commonly used notation, i.e. at a point $m \in M$

$$L_X f(m) \equiv X_m f \equiv X[f](m).$$

Example 4. *In local spherical coordinates in S^2 from Example 3 the Lie derivative of a given function $\alpha : S^2 \to \mathbb{R}$ is expressed as*

$$X\alpha \equiv L_X \alpha = \theta^2 \frac{\partial \alpha}{\partial \theta} - \theta\phi \frac{\partial \alpha}{\partial \phi}.$$

Example 5. Gradient vector field *For any given function $\alpha : M \to \mathbb{R}$ it is possible to construct a vector field using its gradient, i.e.*

$$X = \nabla\alpha = \left(\frac{\partial \alpha}{\partial x_1}, ..., \frac{\partial \alpha}{\partial x_1} \right).$$

Example 6. Dynamics and Lyapunov function *Example: we already saw that for $\dot{x} = f(x)$, $\dot{V} = \frac{\partial V}{\partial x} f(x) \equiv L_f V$, the derivative of V in the direction of the dynamics.*

1.0.2 Lie bracket

Q: Given two vector fields $g_1(x)$ and $g_2(x)$ do their flows commute $\Phi_t^{g_2} \circ \Phi_t^{g_1} = \Phi_t^{g_1} \circ \Phi_t^{g_2}$?

A: In general, no. They bend and twist, unless they are constant vectors.

This is quantified by

$$\Phi_t^{-g_2} \circ \Phi_t^{-g_1} \circ \Phi_t^{g_2} \circ \Phi_t^{g_1}(x_0) = x_0 + t^2[g_1, g_2] + O(t^3)$$

The *Lie bracket* of two vector fields g_1 and g_2 denoted by $[g_1, g_2]$ is a new vector field defined by

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2.$$

or as applied to a function α by

$$[g_1, g_2]\alpha = g_1(g_2\alpha) - g_2(g_1\alpha),$$

Intuition: $g_1\alpha$ denotes the directional derivative of a function α in the direction generated by g_1.

Lie bracket $[g_1, g_2]$ is the directional derivative of a vector field g_2 in the direction generated by g_1.

The bracket has a special role – together with the linear space of vector fields at a point it forms an algebra. More specifically, a vector space V with a bilinear operator $[\cdot, \cdot] : V \times V \to V$ satisfying the following properties

1. Skew-symmetry: $[v\, w] = -[w, v]$ for all $v, w \in V$
2. Jacobi identity:
$$[[v, w], z] + [[z, v], w] + [[w, z], v] = 0,$$
for all $v, w, z \in V$ is a *Lie algebra*.

Example 7. *The unicycle*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 \tag{1}$$

Example 8. *Euclidean space with cross-product:* $(\mathbb{R}^3, [v, w] = v \times w)$, *is a Lie algebra (e.g. applications in* $SO(3)$*)*

Other Examples:

Vector space $(V, [\cdot, \cdot] = 0)$, abelian Lie algebra

Matrix group $(GL_n, [A, B] = AB - BA)$, is a Lie algebra