0.1 Constraints

The configuration space of a mechanical system is denoted by $Q$ and is assumed to be an $n$-dimensional manifold, locally isomorphic to $\mathbb{R}^n$ (we'll say exactly what this means in a future lecture). A configuration is denoted by $q \in Q$.

We first introduce the notion of constraints:

- **holonomic (or geometric):**
  
  $$ h_i(q) = 0, \quad i = 1, \ldots, k $$

  restrict possible motions to a $n - k$ dimensional sub-manifold (think hypersurface embedded in a larger ambient space)

- **linear (Pfaffian) nonholonomic (or kinematic):**
  
  $$ a_i^T(q) \dot{q} = 0, \quad i = 1, \ldots, k, \quad \text{or} \quad A^T(q) \dot{q} = 0 \quad \text{in matrix form} $$

  linear in the velocities

Nonholonomic constraints are not integrable, i.e. it is not possible to find $k$ functions $h_i$ such that

$$ \nabla_q h_i(q) = a_i(q), \quad i = 1, \ldots, k $$

If one can find such functions then the constraint is holonomic, i.e.

$$ \int a_i^T(q(t)) \dot{q}(t) dt = \int \nabla h_i(q(t))^T \dot{q}(t) dt = h_i(q) + c, $$

where $c$ is a constant of integration.

Holonomic constraints are inherently different than nonholonomic. If $a(q)^T \dot{q} = 0$ can be integrated to obtain $h(q) = c$, then the motion is restricted to lie on a level surface (a leaf) of $h$ corresponding to the constant $c$ obtained by the initial condition $c = h(q_0)$. Practically speaking, once the system is on the surface, it cannot escape.

Consider a single constraint $a(q)^T \dot{q} = 0$. When the constraint is nonholonomic the *instantaneous motion* (velocity) is allowed in all directions except for $a(q)$ (i.e. to an $n - 1$-dimensional space). But it could still be possible to reach any configuration in $Q$. So the system will leave the surface.

**Example 1. The unicycle.** The canonical example of a nonholonomic system is the unicycle (a.k.a. the rolling disk). The configuration is $q = (x, y, \theta)$ denoting position $(x, y)$ and orientation $\theta$. There is one constraint, i.e. the unicycle must move in the direction in which it is pointing:

$$ \dot{x} \sin \theta - \dot{y} \cos \theta = 0, \quad \text{or} \quad \frac{\dot{y}}{\dot{x}} = \tan \theta, $$

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We have
\[ a(q) = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{pmatrix}. \]

The feasible velocities are then contained in the null space of \( A(q) = a(q) \), i.e.
\[ \text{null}(a^T(q)) = \text{span} \left\{ \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

This system starts at configuration \( q_0 = (x_0, y_0, \theta_0) \) and can reach any desired final configuration \( q_f = (x_f, y_f, \theta_f) \). The simplest strategy is first to rotate so that the disk points to \((x_f, y_f)\), then move forward until \((x_f, y_f)\) is reached, then turn in place until the orientation reaches \( \theta_f \).

Draw a picture of the motion in the the configuration space.

More generally, let us denote the allowed directions of motion by vectors \( g_j \), i.e.
\[ a_i(q)^T g_j(q) = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n - k \]
or in matrix form
\[ A^T(q)G(q) = 0. \]

The feasible trajectories of the mechanical system are the solutions of
\[ \dot{q} = \sum_{j=1}^{m} g_j(q)v_j = G(q)v, \]
where \( v(t) \in \mathbb{R}^m \), \( m = n - k \), are called reduced velocities or pseudovelocities.

We will be concerned with two classes of models. Kinematic models assume that \( v \) can be directly controlled. Dynamic models require the derivation of another differential equation determining the evolution of \( v \).

For kinematic systems the question of controllability is equivalent to nonholonomy.

### 0.2 Dynamics

How do we obtain \( \dot{x} = f(t, x, u) \) for dynamical systems? We will focus on mechanical systems with equations of motion derived through a Lagrangian approach, which is often sufficient for most systems of interest in robotics.

#### 0.2.1 Holonomic Underactuated Systems

Let \( q \in \mathbb{R}^n \) denote generalized coordinates. Assume that the system has a Lagrangian
\[ L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q), \]
with inertia matrix \( M(q) > 0 \) and potential energy \( V(q) \). The system is subject to external forces \( f_{\text{ext}}(q, \dot{q}) \in \mathbb{R}^n \) and control inputs \( \tau \in \mathbb{R}^m \).
The equations of motion in terms of the Lagrangian (i.e. the Euler-Lagrange equations) are given by
\[ \frac{d}{dt} \nabla \dot{q} \mathbf{L} - \nabla q \mathbf{L} = f_{\text{ext}}(q, \dot{q}) + B(q) \tau, \]
where \( B(q) \in \mathbb{R}^{n \times m} \) is a matrix mapping from \( m \) control inputs to the forces/torques acting on the generalized coordinates \( q \).

This equation is obtained from Lagrange-d’Alembert variational principle
\[ \delta \int_{t_0}^{t_f} L(q, \dot{q}) dt + \int_{t_0}^{t_f} \left[ f_{\text{ext}}(q, \dot{q}) + B(q) \tau \right]^T \delta q(t) = 0. \]

The actual equations take the form
\[ M(q) \ddot{q} + b(q, \dot{q}) = B(q) \tau, \quad (1) \]
where
\[ b(q, \dot{q}) = \dot{M}(q) \dot{q} - \frac{1}{2} \nabla_q (\dot{q}^T M(q) \dot{q}) + \nabla_q V(q) - f_{\text{ext}}(q, \dot{q}). \]

The system is written in control form in terms of the state \( x = (q, \dot{q}) \) as
\[ \dot{x} = f(x) + g(x)u = \begin{pmatrix} \dot{\dot{q}} \\ -M(q)^{-1} b(q, \dot{q}) \end{pmatrix} + \begin{pmatrix} 0 \\ M(q)^{-1} B(q) \end{pmatrix} u \]

\textbf{Example 2.} 2-dof manipulator. Consider a 2 dof-manipulator subject to gravity with the following parameters:

<table>
<thead>
<tr>
<th>Description</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of link #1</td>
<td>( l_1 )</td>
</tr>
<tr>
<td>Length of link #2</td>
<td>( l_2 )</td>
</tr>
<tr>
<td>Distance to COM of link #1</td>
<td>( l_{c1} )</td>
</tr>
<tr>
<td>Distance to COM of link #2</td>
<td>( l_{c2} )</td>
</tr>
<tr>
<td>link #1 mass</td>
<td>( m_1 )</td>
</tr>
<tr>
<td>link #2 mass</td>
<td>( m_2 )</td>
</tr>
<tr>
<td>link #1 inertia</td>
<td>( I_1 )</td>
</tr>
<tr>
<td>link #2 inertia</td>
<td>( I_2 )</td>
</tr>
<tr>
<td>gravity acceleration</td>
<td>( g )</td>
</tr>
</tbody>
</table>

The mass matrix is
\[ M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 [l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos q_2] + I_1 + I_2 & m_2 (l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_2 \\ m_2 (l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_2 & m_2 l_{c2}^2 + I_2 \end{bmatrix}, \]
while the bias term is
\[ b(q, \dot{q}) = \begin{bmatrix} -m_2 l_{c2} \sin(q_2) \dot{q}_2 & -m_2 l_{c2} \sin(q_2) [\dot{q}_1 + \dot{q}_2] \\ m_2 l_{c2} \sin(q_2) \dot{q}_1 & 0 \end{bmatrix} \dot{q} + \begin{bmatrix} [m_1 l_{c1} + m_2 l_1] g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ m_2 l_{c2} g \sin(q_1 + q_2) \end{bmatrix}. \]

For fully actuated manipulator we have \( B(q) = I \). For actuation only at the first joint we have
\[ B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]
Example 3. Simplified model of a boat in 2D, with two rear propellers. The configuration is denoted by \( q = (x, y, \theta) \). The mass matrix is given by

\[
M(q) = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J
\end{bmatrix},
\]

while the bias is

\[
b(q, \dot{q}) = R(\theta)D(\dot{\theta})\dot{q},
\]

where the matrix \( D(\dot{\theta}) \geq 0 \) denotes drag terms and \( R(\theta) \) is the rotation matrix

\[
R(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which transforms forces from body-fixed to spatial frame. The control matrix is

\[
B(q) = R(\theta) \begin{bmatrix}
1 & 1 \\
0 & 0 \\
-r & r
\end{bmatrix},
\]

where the constant \( r > 0 \) denotes the distance between each thruster and central axis.

0.2.2 Nonholonomic Systems

Assume that the system has a Lagrangian

\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^T K(q) \dot{q} - V(q),
\]

with inertia matrix \( K(q) > 0 \) and potential energy \( V(q) \). The system is subject to external forces \( f_{\text{ext}}(q, \dot{q}) \) and control inputs \( \tau \in \mathbb{R}^m \).

The Euler-Lagrange equations take the form

\[
\frac{d}{dt} \nabla_q L - \nabla_{\dot{q}} L = A(q)\lambda + f_{\text{ext}}(q, \dot{q}) + S(q)\tau,
\]

where \( S(q) \in \mathbb{R}^{n \times m} \) is a matrix mapping from \( m \) control inputs to the forces/torques acting on the generalized coordinates \( q \) and where \( \lambda \in \mathbb{R}^k \) is a vector of Lagrange multipliers. The term \( A(q)\lambda \) should be understood as a force which counters any motion in directions spanned by \( A(q) \).

This equation is obtained from the Lagrange-d’Alembert variational principle

\[
\delta \int_{t_0}^{t_f} L(q, \dot{q}) dt + \int_{t_0}^{t_f} \left[ f_{\text{ext}}(q, \dot{q}) + S(q)\tau \right]^T \delta q(t) = 0,
\]

subject to both \( A(q)^T \dot{q} = 0 \) and \( A(q)^T \delta q(t) = 0 \), i.e. the variations are restricted as well.

The actual equations take the form

\[
K(q)\ddot{q} + n(q, \dot{q}) = A(q)\lambda + S(q)\tau, \hspace{1cm} (2)
\]

\[
A^T(q)\dot{q} = 0, \hspace{1cm} (3)
\]
where
\[ n(q, \dot{q}) = K(q)\dot{q} - \frac{1}{2} \nabla_q (q^T K(q) \dot{q}) + \nabla q V(q) \]

The Lagrange multipliers can be eliminated by first noting that
\[ A^T(q)G(q) = 0 \]
and multiplying (??) by \( G^T(q) \) to obtain a reduced set of \( m = n - k \) differential equations
\[ G^T(q)(K(q)\ddot{q} + n(q, \dot{q})) = G^T S(q)\tau. \]

A standard assumption will be that \( \det(G(q)^T S(q)) \neq 0 \) or that all feasible directions are controllable. The final equations are then expressed as
\begin{align*}
\dot{q} &= G(q)v, \quad (4) \\
M(q)\dot{v} + b(q, v) &= B(q)\tau, \quad (5)
\end{align*}

where
\begin{align*}
M(q) &= G^T(q)K(q)G(q) > 0 \\
b(q, v) &= G^T K(q)G(q) + G^T q n(q, G(q)v) \\
B(q) &= G^T(q)S(q)
\end{align*}

using the notation
\[ \dot{G}(q)v = \sum_{i=1}^{m} (\nabla g_i(q)^T v_i)G(q)v. \]

For nonholonomic systems, we would normally assume an isomorphism between pseudo-accelerations \( a = \dot{v} \) and control inputs \( \tau \), i.e. any acceleration \( a \) can be achieved by setting
\[ \tau = B(q)^{-1}(M(q)a + b(q, v)). \]

That is why often in nonholonomic control we take \( a \) as the (virtual) control input, i.e. \( u \equiv a \) and express the control system in terms of the state \( x = (q, v) \)
\[ \dot{x} = f(x) + g(x)u = \left( \begin{array}{c} G(q)v \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ I_m \end{array} \right) u. \]

**Example 4. Unicycle.** The configuration is \( q = (x, y, \theta) \) with mass \( m \), moment of inertia \( J \), driving force \( \tau_1 \), steering force \( \tau_2 \). The general dynamic model
\[ K(q)\ddot{q} + n(q, \ddot{q}) = A(q)\lambda + S(q)\tau, \]
takes the form
\[
\begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
\sin \theta \\
-\cos \theta \\
0
\end{bmatrix} \lambda + \begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix},
\]
We have $G(q) = S(q)$, $G^T(q)S(q) = I_2$, and $G^T(q)B\dot{G}(q) = 0$, from which we obtain the reduced mass matrix and bias

$$M(q) = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}, \quad b(q, \dot{q}) = 0.$$ 

The complete equations of motion are

$$\begin{align*}
\dot{x} &= \cos \theta v_1 \\
\dot{y} &= \sin \theta v_1 \\
\dot{\theta} &= v_2 \\
m\dot{v}_1 &= \tau_1 \\
J\dot{v}_2 &= \tau_2,
\end{align*}$$

which can be put in a standard form, for $\mathbf{x} = (x, y, \theta, v_1, v_2)$

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{\tau}.$$ 

**Example 5. Simple car models.** A common way to model a car for control purposes is to employ the bycycle model, i.e. collapse each pair of wheels to a single wheel at the center of their axle. The generalized coordinates are

$$\mathbf{q} = (x, y, \theta, \phi),$$

where $\phi$ is the *steering angle*. We have the constraints

$$\begin{align*}
\dot{x} \sin \theta - \dot{y} \cos \theta &= 0 & \text{front wheel} \\
\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \dot{\theta} \ell \cos \phi &= 0 & \text{rear wheel}
\end{align*}$$

For the real-wheel drive we have

$$G(q) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ \ell \tan \phi & 0 \\ 0 & 1 \end{bmatrix}$$

while for the front-wheel drive we have

$$G(q) = \begin{bmatrix} \cos \theta \cos \phi & 0 \\ \sin \theta \cos \phi & 0 \\ \ell \sin \phi & 0 \\ 0 & 1 \end{bmatrix}$$

A *kinematic model* is given by

$$\dot{\mathbf{q}} = G(q)\mathbf{u},$$

where the inputs are $u_1$ – rear drive velocity, $u_2$ - steering. A dynamic model includes the dynamics of $\dot{v}$ which is the same as the unicycle.