# EN530.678 Nonlinear Control and Planning in Robotics
# Homework #2

February 14, 2018

Due: February 21, 2018 (before class)

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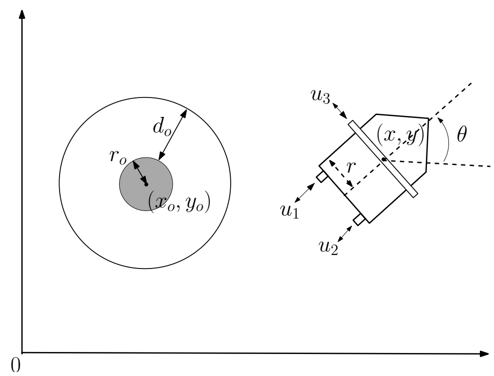


Figure 1: Omnidirectional hovercraft.

1. (Khalil) Euler equations for a rotating rigid spacecraft are given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1,$$
$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2,$$
$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3,$$

where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector $\omega$ along the principal axes, $u_1, u_2, u_3$ are the torque inputs applied about the principal axes, and $J_1, J_2, J_3$ are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable.

(b) Is it asymptotically stable?

(c) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where $k_1, k_2, k_3$ are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

2. (Khalil) Consider the $m$-link robot dynamics

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u,$$

where $q, u \in \mathbb{R}^n$, $M(q)$ is symmetric positive definite. The matrix $C$ has the property that $\dot{M} - 2C$ is skew-symmetric [1] for all $q, \dot{q} \in \mathbb{R}^n$. The term $D\dot{q}$ accounts for viscous damping, where $D$ is positive semidefinite symmetric matrix. The term $g(q)$ is computed according to $g(q) = \nabla P(q)$ where $P(q)$ is the potential energy of the system. Assume that $P(q)$ is positive definite and $g(q) = 0$ has an isolated root at $q = 0$.

(a) with $u = 0$ use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as Lyapunov function to show that the origin $(q = 0, \dot{q} = 0)$ is stable.

(b) with $u = -K_d\dot{q}$, where $K_d$ is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) with $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where $K_p$ and $K_d$ are positive diagonal matrices and $q^*$ is a desired robot position in $R^n$ , show that the point $(q = q^*, \dot{q} = 0)$ is an asymptotically stable equilibrium point.

3. Design of a stabilizing controller for a simple mechanical system and Matlab implementation.

   Consider an omnidirectional hovercraft (Fig. 1) modeled as a fully actuated rigid body in the plane. It has mass $m$ and moment of inertia $J$. It is controlled with three bidirectional thrusters. Two of them are placed in the rear at distance $r$ from the central axis, and the third passes through the body laterally aligned with the center of mass. The hovercraft position is denoted by $p = (x, y)$ and its orientation by $\theta$. The system coordinates are $q = (x, y, \theta)$. The forces produced by each thruster are denoted by $u = (u_1, u_2, u_3)$.

   The equations of motion of the system can be expressed as

$$M\ddot{q} + D\dot{q} = B(q)u,$$

   where

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix}, \; D = \begin{pmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_\theta \end{pmatrix}, \; B(q) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -r & r & 0 \end{pmatrix},$$

   with $d_x, d_y, d_\theta > 0$ defining viscous damping constants.

   **Analytical Problems:**

   (a) Design an exponentially stable controller as a function of the state, i.e. $u = k(q, \dot{q})$, so that the system can stabilizes at the origin $(q, \dot{q}) = (0, 0)$. Prove that your controller is exponentially stable.

   (b) Imagine that there is a disk-like obstacle at position $p_o = (x_o, y_o)$ with radius $r_o$ that the vehicle must avoid. Assume that the vehicle can sense the obstacle if it is within $d_o$ meters of it. Augment your control law with an obstacle avoidance term which applies a "steering" force to the $(x, y)$ degrees of freedom defined by

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{k_o}{d(q)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

[1] a skew-symmetric matrix $S \in \mathbb{R}^{n \times n}$ has the property that $x^T S x = 0$ for all $x \in \mathbb{R}^n$

The force is applied only when the vehicle is heading towards an obstacle, i.e. when the angle between the velocity $(\dot{x}, \dot{y})$ and direction towards obstacle is less than $\pi/2$. Here, $k_o$ is positive constant and $d(q) = \sqrt{(x - x_o)^2 + (y - y_o)^2} - r_o$ is the distance between the vehicle and obstacle. Prove the system is globally asymptotically stable.

**Implementation:**

Choose the following model parameters: $m = 1, J = .1, r = .2$, $D = \text{diag}(.01, .1, .02)$.

(a) Obstacle-free case: implement the controller and simulate the closed-loop system from two initial conditions. In both cases set $q(0) = (3, 2, -\pi/4)$. The first initial condition must be with zero velocity (i.e. $\dot{q}(0) = 0$), while the second with non-zero velocity that you're free to choose.

(b) Obstacle avoidance case: add an obstacle with $r_0 = .25$ at position $p_o = (1, 1)$ and set $d_o = 1$. Design and simulate the obstacle avoidance controller from the two initial conditions specified in a). Generate trajectories for a few different choices of $k_o$ and comment on the effect of this gain.

*An example implementation of a simpler point-mass vehicle stabilization with obstacle avoidance is provided for reference. See file* `hw2_example.m`.

Note: Upload your code and plots as a .zip file using `https://goo.gl/forms/1jxhjYUIlKqYdi7g1`; in addition attach a printout of the code and all plots to your homework solutions.