0.1 Constraints

The configuration space of a mechanical system is denoted by \( Q \) and is assumed to be an \( n \)-dimensional manifold, locally isomorphic to \( \mathbb{R}^n \) (we’ll say exactly what this means in a future lecture). A configuration is denoted by \( q \in Q \).

We first introduce the notion of constraints:

- **holonomic (or geometric):**
  \[ h_i(q) = 0, \quad i = 1, \ldots, k \]
  restrict possible motions to a \( n - k \) dimension sub-manifold (think hypersurface embedded in a larger ambient space)

- **linear (Pfaffian) nonholonomic (or kinematic):**
  \[ a_i^T(q)\dot{q} = 0, \quad i = 1, \ldots, k, \quad \text{or} \quad A^T(q)\dot{q} = 0 \quad \text{in matrix form} \]
  linear in the velocities

Nonholonomic constraints are not integrable, i.e. it is not possible to find \( k \) functions \( h_i \) such that

\[ \nabla_q h_i(q) = a_i(q), \quad i = 1, \ldots, k \]

If one can find such functions then the constraint is holonomic, i.e.

\[ \int a_i^T(q(t))\dot{q}(t)dt = \int \nabla h_i(q(t))^T\dot{q}(t)dt = h_i(q) + c, \]

where \( c \) is a constant of integration.

Holonomic constraints are inherently different than nonholonomic. If \( a(q)^T\dot{q} = 0 \) can be integrated to obtain \( h(q) = c \), then the motion is restricted to lie on a level surface (a leaf) of \( h \) corresponding to the constant \( c \) obtained by the initial condition \( c = h(q_0) \). Practically speaking, once the system is on the surface, it cannot escape.

Consider a single constraint \( a(q)^T\dot{q} = 0 \). When the constraint is nonholonomic the *instantaneous motion* (velocity) is allowed in all directions except for \( a(q) \) (i.e. to an \( n - 1 \)-dimensional space). But it could still be possible to reach any configuration in \( Q \). So the system will leave the surface.

**Example 1. The unicycle.** The canonical example of a nonholonomic system is the unicycle (a.k.a. the rolling disk). The configuration is \( q = (x, y, \theta) \) denoting position \( (x, y) \) and orientation \( \theta \). There is one constraint, i.e. the unicycle must move in the direction in which it is pointing:

\[ \dot{x} \sin \theta - \dot{y} \cos \theta = 0, \quad \text{or} \quad \frac{\dot{y}}{\dot{x}} = \tan \theta, \]
We have
\[
a(q) = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{pmatrix}.
\]
The feasible velocities are then contained in the null space of \( A(q) = a(q) \), i.e.
\[
\text{null}(a^T(q)) = \text{span} \left\{ \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
This system starts at configuration \( q_0 = (x_0, y_0, \theta_0) \) and can reach any desired final configuration \( q_f = (x_f, y_f, \theta_f) \). The simplest strategy is first to rotate so that the disk points to \( (x_f, y_f) \), then move forward until \( (x_f, y_f) \) is reached, then turn in place until the orientation reaches \( \theta_f \).

Draw a picture of the motion in the configuration space.

More generally, let us denote the allowed directions of motion by vectors \( g_j \), i.e.
\[
a_i(q)^T g_j(q) = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n - k
\]
or in matrix form
\[
A^T(q)G(q) = 0.
\]
The feasible trajectories of the mechanical system are the solutions of
\[
\dot{q} = \sum_{j=1}^{m} g_j(q)v_j = G(q)v,
\]
where \( v(t) \in \mathbb{R}^m \), \( m = n - k \), are called reduced velocities or pseudovelocities.

We will be concerned with two classes of models. Kinematic models assume that \( v \) can be directly controlled. Dynamic models require the derivation of another differential equation determining the evolution of \( v \).

For kinematic systems the question of controllability is equivalent to nonholonomy.

## 0.2 Dynamics

How do we obtain \( \dot{x} = f(t, x, u) \) for dynamical systems? We will focus on mechanical systems with equations of motion derived through a Lagrangian approach, which is often sufficient for most systems of interest in robotics.

### 0.2.1 Holonomic Underactuated Systems

Let \( q \in \mathbb{R}^n \) denote generalized coordinates. Assume that the system has a Lagrangian
\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q),
\]
with inertia matrix \( M(q) > 0 \) and potential energy \( V(q) \). The system is subject to external forces \( f_{\text{ext}}(q, \dot{q}) \in \mathbb{R}^n \) and control inputs \( \tau \in \mathbb{R}^m \).
The equations of motion in terms of the Lagrangian (i.e. the Euler-Lagrange equations) are given by
\[ \frac{d}{dt} \nabla_\dot{q} L - \nabla_q L = f_{\text{ext}}(q, \dot{q}) + B(q)\tau, \]
where \( B(q) \in \mathbb{R}^{n \times m} \) is a matrix mapping from \( m \) control inputs to the forces/torques acting on the generalized coordinates \( q \).

This equation is obtained from Lagrange-d’Alembert variational principle
\[ \delta \int_{t_0}^{t_f} L(q, \dot{q}) dt + \int_{t_0}^{t_f} \left[ f_{\text{ext}}(q, \dot{q}) + B(q)\tau \right]^T \delta q(t) = 0. \]

The actual equations take the form
\[ M(q)\ddot{q} + b(q, \dot{q}) = B(q)\tau, \]  
(1)
where
\[ b(q, \dot{q}) = \hat{M}(q)\dot{q} - \frac{1}{2} \nabla_\dot{q}(\dot{q}^T \hat{M}(q)\dot{q}) + \nabla_q V(q) - f_{\text{ext}}(q, \dot{q}). \]

The system is written in control form in terms of the state \( x = (q, \dot{q}) \) as
\[ \dot{x} = f(x) + g(x)u = \begin{pmatrix} \ddot{q} \\ -M(q)^{-1}b(q, \dot{q}) \end{pmatrix} + \begin{pmatrix} 0 \\ M(q)^{-1}B(q) \end{pmatrix} u. \]

**Example 2. 2-dof manipulator.** Consider a 2 dof-manipulator subject to gravity with the following parameters:

<table>
<thead>
<tr>
<th>Description</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of link #1</td>
<td>( l_1 )</td>
</tr>
<tr>
<td>Length of link #2</td>
<td>( l_2 )</td>
</tr>
<tr>
<td>Distance to COM of link #1</td>
<td>( l_{c1} )</td>
</tr>
<tr>
<td>Distance to COM of link #2</td>
<td>( l_{c2} )</td>
</tr>
<tr>
<td>link #1 mass</td>
<td>( m_1 )</td>
</tr>
<tr>
<td>link #2 mass</td>
<td>( m_2 )</td>
</tr>
<tr>
<td>link #1 inertia</td>
<td>( I_1 )</td>
</tr>
<tr>
<td>link #2 inertia</td>
<td>( I_2 )</td>
</tr>
<tr>
<td>gravity acceleration</td>
<td>( g )</td>
</tr>
</tbody>
</table>

The mass matrix is
\[ M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 [l_1^2 + l_2^2 + 2l_1l_2 \cos q_2] + I_1 + I_2 & m_2 (l_{c2}^2 + l_1l_2 \cos q_2) + I_2 \\ m_2 (l_{c2}^2 + l_1l_2 \cos q_2) + I_2 & m_2 l_{c2}^2 + I_2 \end{bmatrix}, \]
while the bias term is
\[ b(q, \dot{q}) = \begin{bmatrix} -m_2 l_1 l_{c2} \sin(q_2) \dot{q}_2 \\ m_2 l_1 l_{c2} \sin(q_2) \dot{q}_1 \end{bmatrix} \dot{q} + \begin{bmatrix} [m_1 l_{c1} + m_2 l_1]g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ m_2 l_{c2} g \sin(q_1 + q_2) \end{bmatrix}. \]

For fully actuated manipulator we have \( B(q) = I \). For actuation only at the first joint we have
\[ B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]
Example 3. Simplified model of a boat in 2D, with two rear propellers. The configuration is denoted by \( q = (x, y, \theta) \). The mass matrix is given by

\[
M(q) = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J
\end{bmatrix}.
\]

while the bias is

\[
b(q, \dot{q}) = R(\theta)D(\dot{q})\dot{q},
\]

where the matrix \( D(\dot{q}) \geq 0 \) denotes drag terms and \( R(\theta) \) is the rotation matrix

\[
R(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which transforms forces from body-fixed to spatial frame. The control matrix is

\[
B(q) = R(\theta) \begin{bmatrix}
1 & 1 \\
0 & 0 \\
-r & r
\end{bmatrix},
\]

where the constant \( r > 0 \) denotes the distance between each thruster and central axis.

0.2.2 Nonholonomic Systems

Assume that the system has a Lagrangian

\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^T K(q) \dot{q} - V(q),
\]

with inertia matrix \( K(q) > 0 \) and potential energy \( V(q) \). The system is subject to external forces \( f_{\text{ext}}(q, \dot{q}) \) and control inputs \( \tau \in \mathbb{R}^m \).

The Euler-Lagrange equations take the form

\[
\frac{d}{dt} \nabla_q L - \nabla_q L = A(q)\lambda + f_{\text{ext}}(q, \dot{q}) + S(q)\tau,
\]

where \( S(q) \in \mathbb{R}^{n \times m} \) is a matrix mapping from \( m \) control inputs to the forces/torques acting on the generalized coordinates \( q \) and where \( \lambda \in \mathbb{R}^k \) is a vector of Lagrange multipliers. The term \( A(q)\lambda \) should be understood as a force which counters any motion in directions spanned by \( A(q) \).

This equation is obtained from the Lagrange-d’Alembert variational principle

\[
\delta \int_{t_0}^{t_f} L(q, \dot{q}) dt + \int_{t_0}^{t_f} [f_{\text{ext}}(q, \dot{q}) + S(q)\tau]^T \delta q(t) = 0,
\]

subject to both \( A(q)^T \dot{q} = 0 \) and \( A(q)^T \delta q(t) = 0 \), i.e. the variations are restricted as well.

The actual equations take the form

\[
K(q)\ddot{q} + n(q, \dot{q}) = A(q)\lambda + S(q)\tau, \quad (2)
\]

\[
A^T(q)\dot{q} = 0, \quad (3)
\]
where
\[ n(q, \dot{q}) = \dot{K}(q)\dot{q} - \frac{1}{2} \nabla_q (q^T K(q) \dot{q}) + \nabla_q V(q) \]

The Lagrange multipliers can be eliminated by first noting that
\[ A^T(q)G(q) = 0 \]
and multiplying \[ G^T(q) \] by \[ G^T(q) \] to obtain a reduced set of \( m = n - k \) differential equations
\[ G^T(q)(K(q)\ddot{q} + n(q, \dot{q})) = G^T S(q)\tau. \]

A standard assumption will be that \( \det(G(q)^T S(q)) \neq 0 \) or that all feasible directions are controllable. The final equations are then expressed as
\[ \dot{q} = G(q)v, \]
\[ M(q)\dot{v} + b(q, v) = B(q)\tau, \]
where
\[ M(q) = G^T(q)K(q)G(q) > 0 \]
\[ b(q, v) = G^T K(q)\dot{G}(q)v + G^T(q)n(q, G(q)v) \]
\[ B(q) = G^T(q)S(q) \]
using the notation
\[ \dot{G}(q)v = \sum_{i=1}^{m} (\nabla g_i(q)^T v_i) G(q)v. \]

For nonholonomic systems, we would normally assume an isomorphism between pseudo-accelerations \( a = \dot{v} \) and control inputs \( \tau \), i.e. any acceleration \( a \) can be achieved by setting
\[ \tau = B(q)^{-1}(M(q)a + b(q, v)). \]

That is why often in nonholonomic control we take \( a \) as the (virtual) control input, i.e. \( u \equiv a \) and express the control system in terms of the state \( x = (q, v) \)
\[ \dot{x} = f(x) + g(x)u = \begin{pmatrix} G(q)v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u. \]

**Example 4. Unicycle.** The configuration is \( q = (x, y, \theta) \) with mass \( m \), moment of inertia \( J \), driving force \( \tau_1 \), steering force \( \tau_2 \). The general dynamic model
\[ K(q)\ddot{q} + n(q, \dot{q}) = A(q)\lambda + S(q)\tau, \]
takes the form
\[ \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ \sin \theta \end{pmatrix} \lambda + \begin{pmatrix} \cos \theta & 0 & 0 \\ \sin \theta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}. \]
We have $G(q) = S(q)$, $G^T(q)S(q) = I_2$, and $G^T(q)B\dot{G}(q) = 0$, from which we obtain the reduced mass matrix and bias

$$M(q) = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}, \quad b(q, \dot{q}) = 0.$$ 

The complete equations of motion are

$$\dot{x} = \cos \theta v_1$$
$$\dot{y} = \sin \theta v_1$$
$$\dot{\theta} = v_2$$
$$m\dot{v}_1 = \tau_1$$
$$J\dot{v}_2 = \tau_2,$$

which can be put in a standard form, for $x = (x, y, \theta, v_1, v_2)$

$$\dot{x} = f(x) + g(x)\tau.$$

**Example 5. Simple car models.** A common way to model a car for control purposes is to employ the bycycle model, i.e. collapse each pair of wheels to a single wheel at the center of their axle. The generalized coordinates are

$$q = (x, y, \theta, \phi),$$

where $\phi$ is the *steering angle*. We have the constraints

$$\dot{x}\sin \theta - \dot{y}\cos \theta = 0 \quad \text{front wheel} \quad (6)$$
$$\dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) - \dot{\theta}\ell \cos \phi = 0 \quad \text{rear wheel} \quad (7)$$

For the real-wheel drive we have

$$G(q) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ \frac{1}{\ell} \tan \phi & 0 \\ 0 & 1 \end{bmatrix}$$

while for the front-wheel drive we have

$$G(q) = \begin{bmatrix} \cos \theta \cos \phi & 0 \\ \sin \theta \cos \phi & 0 \\ \frac{1}{\ell} \sin \phi & 0 \\ 0 & 1 \end{bmatrix}$$

A *kinematic model* is given by

$$\dot{q} = G(q)u,$$

where the inputs are $u_1$ – rear drive velocity, $u_2$ - steering. A dynamic model includes the dynamics of $\dot{v}$ which is the same as the unicycle.