1 Continuous Linear Quadratic Regulator (LQR)

1.1 Finite-Time LQR

Consider a system with dynamics

$$\dot{x} = Ax + Bu$$

which must optimally reach the origin, a task specified by the cost function

$$J = \frac{1}{2} x^T(t_f)P_f x(t_f) + \int_{t_0}^{t_f} \frac{1}{2} \left[ x(t)^T Q(t)x(t) + u(t)^T R(t)u(t) \right] dt,$$

where $P_f$ and $Q$ are symmetric positive semi-definite matrices and $R$ is a symmetric positive definite matrix. Applying the optimality conditions using the Hamiltonian

$$H = \frac{1}{2} (x^T Q x + u^T R u) + \lambda^T (Ax + Bu)$$

we obtain

$$\dot{\lambda} = -Qx - A^T \lambda,$$

while the control is computed according to

$$Ru + B^T \lambda = 0 \quad \Rightarrow \quad u = -R^{-1} B^T \lambda$$

and transversality conditions become

$$\lambda(t_f) = P_f x(t_f).$$

The optimal state $(x(t), \lambda(t))$ then evolves according to the EL equations

$$\begin{pmatrix}
\dot{x} \\
\dot{\lambda}
\end{pmatrix} =
\begin{pmatrix}
A & -BR^{-1}B^T \\
-Q & -A^T
\end{pmatrix}
\begin{pmatrix}
x \\
\lambda
\end{pmatrix},$$

with a final boundary condition $\lambda(t_f) = P_f x(t_f)$. When $A, B, Q, R$ are constant this can be solved for given $y(0) = (x(0), \lambda(0))$ as $y(t) = e^{tM}y(0)$ for instance by computing the Laplace transform of $(sI - M)^{-1}$. This becomes difficult in high-dimensions, but thankfully there’s an easier way.

Kalman showed that the multipliers $\lambda(t)$ are in fact linear function of the states, i.e.

$$\lambda(t) = P(t)x(t).$$
Hence the control can be written according to
\[ u = -R^{-1}B^T P x \equiv K x. \]

The matrix \( P \) can now be computed by noting that
\[ \dot{\lambda} = \dot{P} x + P \dot{x} \]
which is equivalent to
\[ -Q x - A^T P x = \dot{P} x + P A x - P B R^{-1} B^T P x. \]
A solution exists then if we can find a \( P \) which satisfies
\[ \dot{P} = -A^T P - P A + P B R^{-1} B^T P - Q, \quad P(t_f) = P_f \]
This is called the *Riccati ODE* and is integrated from \( t_f \) to \( t_0 \) backwards in time. After \( P(t) \) is found the control is updated according to
\[ u(t) = -R^{-1} B^T P(t) x(t). \]

Note that it turns out that the optimal control \( u \) is in a linear feedback form, i.e. it is a linear function of the state \( x \). This means that we have obtained not only a single optimal control signal from the start state \( x(t_0) \) but also an optimal feedback controller from any state \( x(t) \) for \( t > t_0 \). Therefore, we have completely eliminated the need for an additional controller to physically bring the system to the equilibrium state \( x = 0 \). Furthermore, when the system deviates from the initially computed path e.g. due to disturbances, the same \( P(t) \) computed once in the beginning (at time \( t = t_0 \) can be used from the perturbed state \( x(t) \).

**Stability.** In this context it is instructive to study the stability of the optimal control regarded as a feedback controller \( u = K x \). The closed-loop matrix is
\[ A + BK = A - BR^{-1} B^T P \]
and one should be able to verify that the real parts of its eigenvalues are negative, i.e. that
\[ \text{Real} \{ \text{eig}(A + BK) \} < 0. \]
Here we have assumed that the system \((A, B)\) is controllable.

**Example 1. Linear-quadratic problem.** Consider the system \( \dot{x}(t) = x(t) + u(t) \) with initial condition \( x(0) = x_0 \) and quadratic cost functional
\[ J = \frac{1}{2} \int_0^{t_f} x(t)^2 + u(t)^2 \, dt \]
We have \( A = B = Q = R = 1 \) and the Riccati equation becomes
\[ \dot{P} = -2P + P^2 - 1, \]
with final condition \( P(t_f) = 0 \). The solution can be obtained analytically and is
\[ P(t) = 1 - \sqrt{2} \tanh(\sqrt{2}(t - tf + (\sqrt{2}\tanh(\sqrt{2}/2))/2)) \]
The gain computed with \( t_f = 5 \) is given below
Note: one standard way to get the solution above is to set \( P(t) = \frac{-\ddot{b}(t)}{\dot{b}(t)} \) for some \( b(t) \):

\[
\dot{P} = -\frac{\ddot{b}}{\dot{b}} + \frac{\dot{b}^2}{b^2} = -\frac{\ddot{b}}{\dot{b}} + k^2 \Rightarrow \ddot{b} = -2\dot{b} + b
\]

The solution of the second order linear ODE

\[
\ddot{b} = -2\dot{b} + b, \quad t < t_f, \tag{3}
\]

\[
\dot{b}(t_f) = 0, \; b(t_f) = 1 \tag{4}
\]

is then computed from which we find \( P(t) \).

**Example 2. 2-dim Linear-quadratic problem.** Consider the dynamics

\[
\begin{align*}
\dot{x}_1 &= x_2, \tag{5} \\
\dot{x}_2 &= 2x_1 - x_2 + u
\end{align*}
\]

and cost function

\[
J = \frac{1}{2} \int_0^{t_f} [x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}u^2] dt
\]

We integrate the Riccati ODE numerically using \( t_f = 5 \) and start state \( x(0) = (-4, 4) \). The following matrix \( P(t) \), inputs \( u(t) \) and state histories \( x(t) \) are obtained:
Optimal Cost.  Assume the system is at state $x(t)$.  We can compute the resulting optimal cost from time $t$ to time $t_f$ as follows:

$$ J(t) = \frac{1}{2} x^T(t_f) P_f x(t_f) + \int_t^{t_f} \frac{1}{2} \left[ x^T Q x + u^T R u \right] dt,$$

$$ = \frac{1}{2} x^T(t_f) P_f x(t_f) + \int_t^{t_f} \frac{1}{2} \left[ x(t)^T (Q + K^T R K) x \right] dt,$$

$$ = \frac{1}{2} x^T(t_f) P_f x(t_f) - \int_t^{t_f} \frac{d}{dt} \left( \frac{1}{2} x^T P x \right) dt = \frac{1}{2} x^T(t) P(t)x(t).$$

This will be important for several reasons: stability, cost-to-go, etc...

1.2 Infinite-Time LQR

Consider the state equations $\dot{x} = Ax + Bu$ with cost function

$$ J = \int_{t_0}^{\infty} \frac{1}{2} \left[ x(t)^T Q(t)x(t) + u(t)^T R(t)u(t) \right] dt,$$

The Riccati ODE has the same form but at $t = \infty$ reaches the stationary value

$$ 0 = -A^T P - PA + PBR^{-1} B^T P - Q,$$

which called the algebraic Riccati equation.

The equation can be solved using Matlab

$$ [K, P] = lqr(A, B, Q, R)$$

and $u = K x$ can then be used as the input from state $x$.

1.3 Trajectory Tracking

Consider the problem of not stabilizing to the origin, i.e. $x \to 0$ but tracking a given reference trajectory $x_d(t)$, i.e. $x \to x_d$.  This is often useful when $x_d$ was an optimized trajectory for a complex nonlinear system with constraints, which we cannot reoptimize in real time but can easily track.

One approach is to formulate the error state

$$ e = x - x_d,$$

the control error (assuming we have the control $u_d$ which produced $x_d$)

$$ v = u - u_d,$$

and essentially apply LQR to the dynamics of $e$ subject to “virtual” inputs $v$.  In particular, note that in the general nonlinear case we have

$$ \dot{e} = \dot{x} - \dot{x}_d = f(x, u) - f(x_d, u_d) = f(x_d + e, u_d + v) - f(x_d, u_d) \equiv F(e, v, x_d(t), u_d(t)),$$
or in other words we have obtained a new ODE in for $e$, $v$, and time-varying parameters. In the linear case the we have

$$\dot{e} = Ae + Bv,$$

and so once the optimal $v = Ke$ is computed using standard LQR the actual control $u$ is recovered by

$$u = K(x - x_d) + u_d.$$

Another approach is to directly obtain necessary conditions. In particular, let the cost be defined as

$$J = \frac{1}{2}\|x(t_f) - x_d(t_f)\|^2_{P_f} + \frac{1}{2}\int_0^{t_f} \left\{ \|x(t) - x_d(t)\|^2_{Q(t)} + \|u(t)\|^2_{R(t)} \right\} dt$$

The Hamiltonian is

$$H = \frac{1}{2}\|x(t) - x_d(t)\|^2_{Q(t)} + \frac{1}{2}\|u(t)\|^2_{R(t)} + \lambda^T(Ax + Bu)$$

and the necessary conditions become

$$\dot{\lambda} = -Q(x - x_d) - A^T\lambda,$$

and

$$u = -R^{-1}B^T\lambda,$$

while the transversality conditions are

$$\lambda(t_f) = P_f(x(t_f) - x_d(t_f)).$$

In order to derive a control law, we follow the same reasoning as in the regular case and assume that the multiplier is of the form

$$\lambda = Px + s$$

We will now attempt to derive expressions for $P$ and $s$ that satisfy the necessary conditions. Differentiating we have

$$\dot{\lambda} = \dot{P}x + P\dot{x} + \dot{s},$$

which is equivalent to

$$-Qx + Qx_d - A^T(Px + s) = \dot{P}x + (PA - PBR^{-1}B^TP)x - PBR^{-1}B^Ts + \dot{s}$$

or

$$(-A^TP - PA - Q + PBR^{-1}B^TP - \dot{P})x + (\dot{s} + A^TS - PBR^{-1}B^Ts - Qx_d) = 0.$$
Example 3. **Non-zero signal tracking.** Consider the dynamics

\begin{align}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= 2x_1 - x_2 + u
\end{align}

and cost function

\[ J = 4(x_1 - 1)^2 + \frac{1}{2} \int_0^{t_f} [2(x_1 - 1)^2 + 0.005u^2] dt \]

which corresponds to driving only the \( x_1 \) coordinate to 1 while minimizing control effort. This is a simplified version of the more general condition \( x(t) \to x_d(t) \), where \( x_{d1} = 1 \) is constant. We integrate the Riccati ODE numerically using \( t_f = 5 \) to obtain the following matrix \( P(t) \), inputs \( u(t) \) and state histories \( x(t) \):