1 Continuous Systems with Terminal Constraints

Consider the cost

\[ J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt, \]  

subject to \( q \) constraints

\[ \psi(x(t_f), t_f) = 0 \]

and the dynamics

\[ \dot{x}(t) = f(x(t), u(t), t), \quad t_0 \text{ and } x(t_0) \text{ are given.} \]

It will be useful to employ the shorthand notation \( f(t) \equiv f(x(t), u(t), t) \), etc...

Sometimes, \( f \) (or any other function) could also be without arguments, i.e. \( f \equiv f(x(t), u(t), t) \).

Before we obtain the necessary conditions for optimality, let’s see how to deal with variations of terms in the cost defined at \( t_f \). Assume that our functional includes a term \( h(x(t_f), t_f) \), i.e.

\[ J = h(x(t_f), t_f) + \int_{t_0}^{t_f} \ldots dt, \]

where \( \ldots \) represent any other terms. We can express this as

\[ J = h(x(t_0), t_0) + \int_{t_0}^{t_f} \frac{d}{dt} h(x(t), t) + \ldots dt, \]

which is equivalent to

\[ J = h(x(t_0), t_0) + \int_{t_0}^{t_f} \nabla_x h(x(t), t)^T \dot{x}(t) + \partial_t h(x(t), t) + \ldots dt. \]

Then using integration by parts and the definition of \( \delta x_f = \delta x(t_f) + \dot{x} \delta t_f \) we have

\[ \delta J = \nabla_x h(t_f)^T \delta x_f + [\nabla_x h(t_f)^T \dot{x}(t_f) + \partial_t h(t_f)] - \nabla_x h(t_f)^T \dot{x}(t_f) [\delta t_f \] 

\[ + \int_{t_0}^{t_f} \nabla_x [\nabla_x h(t)^T \dot{x}(t) + \partial_t h(t)] - \frac{d}{dt} \nabla_x [\nabla_x h(t)^T \dot{x}(t) + \partial_t h(t)] + \delta(\ldots) dt \] 

After applying the derivatives under the integral, all terms there cancel and we end up with:

\[ \delta J = \nabla_x h(x(t_f), t_f)^T \delta x_f + \partial_t h(x(t_f), t_f) \delta t_f + \int_{t_0}^{t_f} + \delta(\ldots) dt \]
To obtain the necessary conditions for the original problem (1), form the augmented cost

\[ J_a = \phi(t_f) + \nu^T \psi(t_f) + \int_{t_0}^{t_f} \{ L(x,u,t) + \lambda^T(t)[f(x,u,t) - \dot{x}] \} \, dt. \]

Let \( \Phi = \phi + \nu^T \psi \) and define the Hamiltonian \( H \) by

\[ H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t). \]

Taking variations with respect to all variables including final time \( t_f \) (and using the relationship (3)) we obtain

\[ \delta J_a = \left( (\partial_t \Phi + L) \delta t_f + \partial_x \Phi \cdot \delta x_f \right)_{t=t_f} + \int_{t_0}^{t_f} \left( \partial_x H \cdot \delta x + \partial_u H \cdot \delta u - \lambda \delta \dot{x} \right) \, dt. \]

Integrating by parts and using the relationship

\[ \delta x_f = \delta x(t_f) + \dot{x} \delta t_f, \]

we obtain

\[ \delta J_a = \left( [\partial_t \Phi + L + \lambda^T \dot{x}] \delta t_f + [\partial_x \Phi - \lambda^T] \delta x_f \right)_{t=t_f} + \int_{t_0}^{t_f} \left[ (\partial_x H + \lambda^T) \delta x + \partial_u H \cdot \delta u \right] \, dt. \]

Since all variations are arbitrary and independent the necessary conditions become

\[ \dot{x} = f(x,u,t) \] (8)
\[ \dot{\lambda} = -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L \] (9)
\[ \nabla_u H = \nabla_u f \cdot \lambda + \nabla_u L = 0 \] (10)
\[ \lambda(t_f) = \nabla_x \phi(x(t_f),t_f) + \nabla_x \psi(x(t_f),t_f) \cdot \nu, \] (11)
\[ \left( \partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f \right)_{t=t_f} = 0, \] (12)

where

\[ \frac{d\Phi}{dt} = \partial_t \Phi + \partial_x \Phi \cdot \dot{x}. \]

After substituting the expression for \( \lambda(t_f) \) the necessary conditions are summarized according to:

When the final time \( t_f \) is fixed the last relationship (12) can be dropped.
Hamiltonian conservation. Note that whenever the Hamiltonian does not depend on time (that is when \( f \) and \( L \) do not depend on time)

\[
\partial_t H(x, u, \lambda, t) = 0
\]

then \( H \) is a conserved quantity along optimal trajectories \( x^*(t), u^*(t), \lambda^*(t) \), i.e. we have that

\[
\dot{H}(x, u, \lambda, t) = \partial_x H \cdot \dot{x} + \partial_u H \cdot \dot{u} + \partial_{\lambda} H \cdot \dot{\lambda} + \partial_t H = 0
\]

(13)

Therefore, in this case we have \( H(t) = \text{const} \) for all \( t \in [t_0, t_f] \). Furthermore, in the special case when \( \partial_t \phi = 0 \) and \( \partial_t \psi = 0 \) the last condition \( \text{(12)} \) reduces to \( H(t) = 0 \).

Minimum-time problems. For minimum-time problems we have \( \phi = 0 \) and \( L = 1 \) so that condition \( \text{(12)} \) reduces to

\[
\left( \nu^T [\partial_t \psi + \nabla_x \psi^T \cdot f] + 1 \right)_{t=t_f} = 0,
\]

which can be used along with the constraint \( \psi(x(t_f), t_f) = 0 \) to determine the multipliers \( \nu \) and final time \( t_f \).

Solution Methods

We are faced with solving the differential equations for \( t = [t_0, t_f] \):

\[
\text{Euler-Lagrange (EL):} \quad \left( \begin{array}{c} \dot{x} \\ \dot{\lambda} \end{array} \right) = \left( \begin{array}{c} f(x, u, t) \\ -\nabla_x H \end{array} \right)
\]

(15)

where \( u(t) \) is computed by minimizing \( H \) which corresponds to the condition

\[
\text{Control optimization:} \quad \nabla_u H = 0,
\]

which we assume can be solved and that \( u(t) \) is then expressed as a function of \( x(t) \) and \( \lambda(t) \), subject to the boundary constraints

\[
\psi(x(t_f), t_f) = 0
\]

\[
\lambda(t_f) = \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu,
\]

(16)

Transversality Conditions (TC):

\[
\left( \partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f \right)_{t=t_f} = 0,
\]

The following solution methods are applicable based on whether EL can be integrated in closed-form and whether TC can be solved in closed form:

- general: two-point boundary value problem (BVP) works with any EL and TC, the conditions are satisfied using a numerical “collocation” procedure
- EL integrable: pick \( \lambda(0) \) integrate from \( t_0 \) to \( t_f \) and solve TC as an implicit equality for the unknown \( (\lambda(0), \nu) \). When final time \( t_f \) is free then solve for \( (\lambda(0), \nu, t_f) \).
- EL integrable and TC solvable: closed-form solution.
Example 1. Minimum Control Effort Landing
Consider a second order system with state \( x = (p, v) \in \mathbb{R}^4 \) where \( p \in \mathbb{R}^2 \) is the position and \( v \in \mathbb{R}^2 \) is the velocity. The system has a \textit{double integrator dynamics} given by

\[
\begin{pmatrix}
\dot{p} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
v \\
u
\end{pmatrix},
\]

where \( u \in \mathbb{R}^2 \) is the acceleration control input. The system starts with known initial state \( x_0 = (p_0, v_0) \) and must “land” with a prescribed velocity \( v_f \) somewhere on a unit circle centered at the origin, i.e. the final configuration must satisfy \( \psi(x(t_f)) = 0 \), where

\[
\psi(x) = p^T p - 1.
\]

The objective function is the control effort given by

\[
L(x,u) = \frac{1}{2} \|u\|^2
\]

We start with the Hamiltonian, and the multipliers \( \lambda = (\lambda_p, \lambda_v) \)

\[
H = \frac{1}{2} u^T u + \lambda_p^T v + \lambda_v^T u,
\]

We have

\[
\dot{\lambda} = -\nabla_x H \quad \Rightarrow \quad \dot{\lambda}_p = 0, \quad \dot{\lambda}_v = -\lambda_p
\]

\[
\nabla_u H = 0 \quad \Rightarrow \quad u = -\lambda_v,
\]

from which we get

\[
\ddot{u} = -\ddot{\lambda}_v = \dot{\lambda}_p = 0,
\]

which means that the path \( p(t) \) is a cubic spline that can be written according to

\[
p(t_0 + t) = c_3 t^3 + c_2 t^2 + v_0 t + p_0,
\]

while the velocity is

\[
v(t_0 + t) = 3c_3 t^2 + 2c_2 t + v_0.
\]

Now from

\[
\lambda_p(t_f) = \nabla_p \psi(x(t_f)) \nu = 2p(t_f) \nu.
\]

Note that above since the velocity is not present in the terminal constraint \( \psi \), then there is no additional condition on \( \lambda_v(t_f) \).

Now considering that \( \lambda_p(t_f) = \dot{u}(t_f) = 6c_3 \) the above is equivalent to

\[
6c_3 = 2p(t_f) \nu.
\]

Finally, assuming \( t_f \) is given we can solve for \( \nu, c_2, c_3 \) (5 unknowns) the implicit equations (5 equations):

\[
\begin{align*}
6c_3 - 2p(t_f) \nu &= 0, \\
p(t_f)^T p(t_f) - 1 &= 0, \\
v(t_f) - v_f &= 0,
\end{align*}
\]

(19) (20) (21)
where \( p(t_f) \) and \( v(t_f) \) are given by (17) and (18). Note that it is necessary that \( \nu \neq 0 \) to ensure that the constraint is satisfied.

Examples of the resulting trajectories from randomly initialized states are given. In all examples we have \( v_f = (0, 0) \)

![Graph showing trajectories](image)

**Example 2.** Example: Zermelo’s problem (Bryson §2.7) Consider a ship with dynamics

\[
\begin{align*}
\dot{x} &= V \cos \theta + u(x, y) \\
\dot{y} &= V \sin \theta + v(x, y),
\end{align*}
\]

where \((x, y)\) is the position, \(V\) is a constant velocity, \(\theta\) is the heading angle input and \(u\) and \(v\) denote velocity due to currents. The goal is to travel between points \(A\) and \(B\) in minimum time.

The Hamiltonian is

\[
H = \lambda_x (V \cos \theta + u) + \lambda_y (V \sin \theta + v) + 1.
\]

The Euler-Lagrange equations are

\[
\begin{align*}
\dot{\lambda}_x &= -\partial_x H = -\lambda_x \partial_x u - \lambda_y \partial_x v \\
\dot{\lambda}_y &= -\partial_y H = -\lambda_x \partial_y u - \lambda_y \partial_y v \\
0 &= \partial_\theta H = V (\lambda_x \sin \theta + \lambda_y \cos \theta) \Rightarrow \tan \theta = \frac{\lambda_y}{\lambda_x}
\end{align*}
\]

Since this is a minimum-time problem we have \( H = 0 \) and from (26) that

\[
\lambda_x = \frac{-\cos \theta}{V + u \cos \theta + v \sin \theta}, \quad \lambda_y = \frac{-\sin \theta}{V + u \cos \theta + v \sin \theta}
\]

This leads to

\[
\dot{\theta} = \sin^2 \theta \partial_x v + \sin \theta \cos \theta (\partial_x u - \partial_y v) - \cos^2 \theta \partial_y u
\]

Now, in order to reach \(B\) one has to select the start angle \(\theta_A\) and the final time \(t_f\).
Special Case. For the special case when
\[ u = -V \frac{y}{h}, \quad v = 0 \]
consider starting at \((x_0, y_0)\) with the goal to reach the origin \((0, 0)\). We have
\[ \dot{\lambda}_x = 0 \implies \lambda_x = \text{const} \]
and therefore
\[ \lambda_x = \frac{-\cos \theta}{V - V \frac{y}{h} \cos \theta} = \frac{-\cos \theta_f}{V} = -\text{const} \implies \cos \theta = \frac{\cos \theta_f}{1 + \frac{y}{h} \cos \theta_f} \]
In the above, it turned out that it is convenient to work in terms of \(\theta_f\) rather than \(t_f\). The solution can be obtained analytically as
\[ x = \frac{h}{2} \left[ \sec \theta_f (\tan \theta_f - \tan \theta) - \tan \theta (\sec \theta_f - \sec \theta) + \log \frac{\tan \theta_f + \sec \theta_f}{\tan \theta + \sec \theta} \right], \quad (27) \]
\[ y = h (\sec \theta - \sec \theta_f), \quad (28) \]
from which one can compute the initial angle \(\theta\) and final angles \(\theta_f\) to achieve given final position \((x, y)\).

The computed path with initial conditions given by \(x_0 = 3.66\) and \(y_0 = -1.86\) with \(h = 1\), \(V = .3\) are given below.

Example 3. Minimum Control Effort Landing with Optimal Time Consider the minimum control effort landing with free final time \(t_f\) and a cost function given by
\[ L(x, u) = b + \frac{1}{2} \|u\|^2, \]
for some constant \(b > 0\) which controls the balance between penalizing total time and total control effort.
We need to add the third transversality condition from (16)

\[ \partial_t \phi(t_f) + H(t_f) = 0, \]

which in our case is

\[ b - \frac{1}{2} \| u(t_f) \|^2 + \dot{u}(t_f)^T v(t_f) = 0 \]

This can be solved along with the other five conditions to obtain the unknowns \( c_2, c_3, \nu, t_f \). Plots of computed trajectories with varying \( b \) are given below.