1 Variations of functions

We are interested in solving optimal control problems such as

$$\min J(x(\cdot), u(\cdot), t_f) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt$$

subject to $\dot{x} = f(x, u, t)$ and other constraints. The cost $J$ is called a functional, i.e. it is a function of functions since the trajectories $x(\cdot)$ and $u(\cdot)$ are functions of time. It is possible to optimize a functional in a similar way we optimize a regular function. In particular, there is a functional analog to the necessary conditions for a minimum of a function $g$ given by

$$\nabla g = 0 \iff \text{variation of a functional } \delta J = 0$$

Next, define the change in a functional, after varying $x(t)$ by $\delta x(t)$ at each $t$, by

$$\Delta J(x(\cdot), \delta x(\cdot)) = J(x(\cdot) + \delta x(\cdot)) - J(x(\cdot))$$

The variation $\delta J(x(\cdot), \delta x(\cdot))$ is a linear function of $\delta x(\cdot)$ defined by the following relationship

$$\Delta J(x(\cdot), \delta x(\cdot)) = \delta J(x(\cdot), \delta x(\cdot)) + o(\|\delta x\|),$$

where the small-o notation was used. For a positive integer $p$ and a function $h : \mathbb{R}^n \to \mathbb{R}^m$ we have

$$h(x) = o(\|x\|^p),$$

if $\lim_{k \to \infty} \frac{h(x_k)}{\|x_k\|^p} = 0$ for all sequences $x_k$ such that $x_k \to 0$ and $x_k \neq 0$ for all $k$.

In other words, roughly speaking, if $h(x)$ denotes the the second and higher order terms in $\Delta J$, then $h(x)$ has second and higher-order multiples of $\delta x$ and thus $h(x)/\|\delta x\|$ is at least proportional to $\delta x$ which tends to zero as $\delta x \to 0$. 

Similarly to standard function optimization, we can use the argument that at an optimum \(x^*(\cdot)\) we have
\[
\Delta J(x^*(\cdot), \delta x(\cdot)) \geq 0
\]
since any change of the cost away from optimum must be positive. Taking \(\delta x \to 0\) this implies that
\[
\delta J(x^*(\cdot), \delta x(\cdot)) \geq 0.
\]
But this must also hold for variations in the direction \(-\delta x(\cdot)\), i.e.
\[
\delta J(x^*(\cdot), -\delta x(\cdot)) \geq 0 \implies \delta J(x^*(\cdot), \delta x(\cdot)) \leq 0,
\]
by the linearity of \(\delta J\). Both conditions can only be true when
\[
\delta J(x^*(\cdot), \delta x(\cdot)) = 0,
\]
for any \(\delta x(\cdot)\).

2 The Euler-Lagrange Equations

Consider the cost function
\[
J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t)) dt.
\]
Its variation becomes
\[
\delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) \delta x + g_{\dot{x}}(x, \dot{x}) \delta \dot{x} \right] dt,
\]
since the variation corresponds to taking first-order terms (i.e. derivatives) of the functions at time \(t\). Integrate by parts (recall \(\int u \dot{v} = uv - \int \dot{u} v\)) to get
\[
\delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) \delta x - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \delta \dot{x} \right] dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f)) \delta x(t_f)
\]
Fixed boundary conditions. If \(x(t_0)\) and \(x(t_f)\) are given then \(\delta x(0) = \delta x(t_f) = 0\) and
\[
\delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \right] \delta \dot{x}(t) dt
\]
Since \(\delta x(t)\) are arbitrary and independent then \(\delta J = 0\) only when
\[
g_x(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) = 0,
\]
which are called the Euler-Lagrange equations (EL).

Free boundary conditions. When \(x(t_f)\) is not fixed, in addition to the EL equation, the following must hold
\[
g_{\dot{x}}(x(t_f), \dot{x}(t_f)) = 0.
\]
2.1 Example: shortest path curve

Consider a one-dimensional problem with state \( x(t) \) and the cost function

\[
J = \int_{t_0}^{t_f} \sqrt{1 + x'^2} \, dt,
\]

which in fact corresponds to the length of a curve in the \((x,t)\) plane. The goal is to compute the shortest length curve between two given points \((x_0, t_0)\) and \((x_f, t_f)\).

Applying the EL equations we get

\[
\frac{d}{dt} \frac{x'}{(1 + x'^2)^{1/2}} = 0 \quad \Leftrightarrow \quad \frac{x'}{(1 + x'^2)^{3/2}} = 0,
\]

which is satisfied when \( \ddot{x} = 0 \) or when

\[
x(t) = c_1 t + c_0,
\]

i.e. when \( x(t) \) is a straight line. Let \( t_0 = 0 \) and \( t_f = 1 \). It is easy to see that \( c_0 = x_0 \) and \( c_1 = x_f - x_0 \).

2.2 Particle in 3-D

Let \( g(x, \dot{x}, t) = \frac{1}{2} m ||\dot{x}||^2 - V(x) \), where \( m \) denotes the mass and \( V \) denotes the potential energy of a particle with position \( x \in \mathbb{R}^3 \). The function \( g \) is actually the Lagrangian of the particle and the EL equations lead to

\[
m\ddot{x} = -\nabla_x V,
\]

which is simply Newton’s law. This is one of the simplest examples that illustrates that Lagrangian mechanics can be considered as a special case of optimal control.

3 Free final-time

When the final time \( t_f \) is allowed to vary, the variation of \( J \) is expressed as

\[
\delta J = \int_{t_0}^{t_f} \left[ \frac{d}{dt} g_x(x^*, \dot{x}^*, t) \delta x - g_x(x^*, \dot{x}^*, t) \delta x \right] dt + g_x(x^*(t_f), \dot{x}^*(t_f), t) \delta x(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f,
\]

where we employ the spatially-optimized trajectory \( x^*(t) \) instead of just \( x(t) \) to signify that after finding the optimal path without considering \( \delta t_f \), then variations of \( t_f \) infinitesimally contribute to the cost by the term

\[
g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f
\]

which is simply the cost at the last points multiplied by the time variations: think of it as a first-order approximation to the integral \( \int_{t_f}^{t_f + \delta t_f} g(x^*, \dot{x}^*, t) dt \) which is what must be added to the cost when varying time.
Define the total space-time variation $\delta x_f$ by

$$\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f,$$

i.e. this variation combines the effects of varying the final state by keeping time $t_f$ fixed, and then by varying $t_f$ (See figure above).

We have

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t) \delta x - \frac{d}{dt} g_x(x^*, \dot{x}^*, t) \delta x \right] dt$$

$$+ \left( g_x(x^*, \dot{x}^*, t_f) \delta x_f + [g(x^*, \dot{x}^*, t_f) - g_x(x^*, \dot{x}^*, t_f)] \dot{x} \delta t_f \right)_{t=t_f}.$$

**Unrelated $t_f$ and $x(t_f)$.** We next consider the case when $t_f$ and $x(t_f)$ are unrelated in which we have

$$g_x(t_f) = 0, \quad [g(t_f) - g_x(t_f) \dot{x}] \delta t_f = 0 \quad \Rightarrow \quad g(t_f) = 0$$

**Function $\Theta(t_f)$.** We next consider the case when the boundary constraint is given by

$$x(t_f) = \Theta(t_f)$$

Variations $\delta t_f$ and $\delta x_f$ are related by

$$\delta x_f = \frac{d\Theta}{dt} \delta t_f$$

and hence the necessary conditions become

$$g_x(t_f) \left[ \frac{d\Theta}{dt} - \dot{x}^* \right]_{t=t_f} + g(t_f) = 0,$$

which are called *trasversality conditions*.

## 4 Differential Constraints

Consider the optimization of

$$J = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$
where $x \in \mathbb{R}^{n+m}$ subject to

\[ f(x(t), \dot{x}(t), t) = 0, \]

where $f = (f_1, \ldots, f_n)$ are $n$ constraints, and $x(t_f)$ and $t_f$ are fixed. To obtain the necessary conditions, we define the Lagrangian multipliers $\lambda : [t_0, t_f] \to \mathbb{R}^n$ and the augmented cost

\[ J_a = \int_{t_0}^{t_f} \left\{ g(x, \dot{x}, t) + \lambda^T f(x, \dot{x}, t) \right\} \, dt \]

Taking variations

\[ J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x] \delta x(t) + [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta \dot{x} + \delta \lambda^T f \right\} \, dt \]

Integrating by parts we get

\[ J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x - \frac{d}{dt} [g_{\dot{x}} + \lambda^T f_{\dot{x}}]] \delta x + \delta \lambda^T f \right\} \, dt. \]

If we define the augmented cost $g_a$ by

\[ g_a = g + \lambda^T f \]

the Euler-Lagrange equations become

\[ \frac{\partial g_a}{\partial x} - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{x}} = 0 \]

which along with the constraint

\[ f(x, \dot{x}, t) = 0 \]

constitute the necessary conditions.

5 General Boundary Constraints

Let $x \in \mathbb{R}^n$ and consider the optimization of

\[ J(x(\cdot), t_f) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) \, dt \]

subject to the free final time $t_f$ and general boundary conditions

\[ \psi(x(t_f), t_f) = 0, \]

where $\psi$ is a vector of $m$ functions. To obtain the necessary conditions define the augmented cost

\[ J_a = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) \, dt, \]

where $\nu \in \mathbb{R}^m$. Let

\[ w(x(t_f), \nu, t_f) = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f) \]
Taking variations

\[ \delta J_a = w_x(t_f) \delta x_f + w_{t_f}(t_f) \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x \cdot \delta x(t) + g_\dot{x} \cdot \delta \dot{x}(t)] dt + g(t_f) \delta t_f, \]

where the last term is due to cost accrued from final time variations. Using integration by parts as well as the total variation definition \( \delta x_f = \delta x(t_f) + \dot{x}(t_f) \delta t_f \) we have

\[
\int_{t_0}^{t_f} g_\dot{x} \cdot \delta \dot{x}(t) dt = g_\dot{x}(t_f) \delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt} g_\dot{x} \cdot \delta x(t)
\]

which results in

\[
\delta J_a = [w_x(t_f) + g_\dot{x}] \delta x_f + [w_{t_f}(t_f) + g(t_f) - g_\dot{x}(t_f) \dot{x}(t_f)] \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x - \frac{d}{dt} g_\dot{x}] \delta x(t) dt,
\]

The necessary conditions require that \( \delta J_a = 0 \) for arbitrary \( \delta x(t), \delta \nu \) which is only possible if the following necessary conditions hold:

\[
\nabla_x w(x(t_f), \nu, t_f) + \nabla_\dot{x} g(x(t_f), \dot{x}(t_f), t_f) = 0, \tag{1}
\]

\[
\frac{\partial}{\partial t_f} w(x(t_f), \nu, t_f) + g(x(t_f), \dot{x}(t_f), t_f) - \nabla_\dot{x} g(x(t_f), \dot{x}(t_f), t_f)^T \dot{x}(t_f) = 0, \tag{2}
\]

\[
\psi(x(t_f), t_f) = 0 \tag{3}
\]

\[
\nabla_x g(x(t), \dot{x}(t), t) - \frac{d}{dt} \nabla_\dot{x} g(x(t), \dot{x}(t), t) = 0, \quad t \in (t_0, t_f). \tag{4}
\]