1. Consider the minimization of
\[ J = \frac{1}{2}x^2(1) + \int_0^1 \frac{1}{2}[x(t)u(t)]^2 dt, \]
subject to the nonlinear dynamics
\[ \dot{x} = xu, \quad x(0) = 1. \]
Derive an optimal feedback control using the HJB equation. In the process, show that the HJB partial differential equation has a solution that is a quadratic function in \( x \).

2. Consider the LQR problem with known disturbance \( w(t) \) which requires the minimization of
\[ J = \frac{1}{2}x(t_f)^T P_f x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x(t)^T Q(t)x(t) + u(t)^T R(t)u(t)] dt, \]
subject to
\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t). \]

a) Using the HJB equation, show that a possible value function has the form
\[ V(x(t), t) = \frac{1}{2}x(t)^T P(t)x(t) + b(t)^T x(t) + c(t) \]
and show that the associate optimal control is
\[ u(t) = K(t)x(t) + k(t), \]
where \( K(t) \) and \( k(t) \) are the feedback gain matrix and feedforward term, respectively. Derive the differential equations for \( \dot{P} \), \( \dot{b} \), \( \dot{c} \) and their boundary conditions which satisfy the HJB equation.

b) Do the discrete-time equivalent of Part a).

3. Direct Shooting: An example of a direct shooting method is provided with an example application to a simple car model (car_shooting.m) that exploits the least-squares structure of the cost function. You are required to apply the same method to a new system of your choice, or to the two-link arm example with discrete dynamic function \( f_k \) given in arm_sim.m.
Details: This problem is implemented using the Gauss-Newton (GN) least-squares method for optimizing over the controls $\xi \triangleq u_{0:N-1}$. Using the dynamics, each state can be expressed as a function of the controls $\xi$ which is encoded through the functions $x_k = \psi_k(\xi)$ for $k = 0, \ldots, N$. The cost is then expressed as $J(\xi) = \frac{1}{2} g(\xi)^T g(\xi)$, where $g(\xi)$ is given by

$$g(\xi) = \begin{bmatrix}
\sqrt{R_0} (u_0 - u_d) \\
\sqrt{Q_1} (\psi_1(\xi) - x_d) \\
\sqrt{R_1} (u_1 - u_d) \\
\vdots \\
\sqrt{Q_{N-1}} (\psi_{N-1}(\xi) - x_d) \\
\sqrt{R_{N-1}} (u_{N-1} - u_d) \\
\sqrt{Q_f} (\psi_N(\xi) - x_f)
\end{bmatrix},$$

for some desired control $u_d$, desired state $x_d$, and desired final state $x_f$. Since $R_k > 0$ the Jacobian $\partial g(\xi)$ is guaranteed to be full rank and one can apply a GN iterative method directly to update $\xi \rightarrow \xi + \delta \xi$ where $\delta \xi = -(\partial g^T \partial g)^{-1} \partial g^T g$. In addition, the Jacobian has a lower-triangular structure that can be exploited in the Cholesky GN solution.

4. Direct Collocation: Implement a nonlinear programming strategy using direct transcription/collocation (as explained in the notes) to one of the two provided models (car or arm), or to a model of your choice. This can be accomplished by defining the cost and constraints and finding a solution using Matlab \texttt{fmincon}. See example \texttt{trajopt_sqp_car.m}. Obstacles should be added as inequality constraints.

Note: upload your code to the File upload link and in addition attach a printout of your code and plots (annotated if necessary) to your homework solutions.