EN530.603 Applied Optimal Control
Homework #5
October 24, 2017

Due: October 31, 2017 (before class)

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1. Consider the minimization of

$$J = \frac{1}{2}x(1)^2 + \int_0^1 \frac{1}{2}[x(t)u(t)]^2 dt,$$

subject to the nonlinear dynamics

$$\dot{x} = xu, \qquad x(0) = 1.$$

Derive an optimal feedback control using the HJB equation. In the process, show that the HJB partial differential equation has a solution that is a quadratic function in x.

2. Consider the LQR problem with known disturbance $w(t)$ which requires the minimization of

$$J = \frac{1}{2}x(t_f)^T P_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [x(t)^T Q(t)x(t) + u(t)^T R(t)u(t)]dt,$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t).$$

a) Using the HJB equation, show that a possible value function has the form

$$V(x(t), t) = \frac{1}{2}x(t)^T P(t)x(t) + b(t)^T x(t) + c(t)$$

and show that the associate optimal control is

$$u(t) = K(t)x(t) + k(t),$$

where $K(t)$ and $k(t)$ are the feedback gain matrix and feedforward term, respectively. Derive the differential equations for $\dot{P}, \dot{b}, \dot{c}$ and their boundary conditions which satisfy the HJB equation.

b) Do the discrete-time equivalent of Part a).

3. *Direct Shooting*: An example of a direct shooting method is provided with an example application to a simple car model (`car_shooting.m`) that exploits the least-squares structure of the cost function. You are required to apply the same method to a new system of your choice, or to the two-link arm example with discrete dynamic function f_k given in `arm_sim.m`.

Details: This problem is implemented using the Gauss-Newton (GN) least-squares method for optimizing over the controls $\xi \triangleq u_{0:N-1}$. Using the dynamics, each state can be expressed as a function of the controls ξ which is encoded through the functions $x_k = \psi_k(\xi)$ for $k = 0, \ldots, N$. The cost is then expressed as $J(\xi) = \frac{1}{2} g(\xi)^T g(\xi)$, where $g(\xi)$ is given by

$$g(\xi) = \begin{bmatrix} \sqrt{R_0}\,(u_0 - u_d) \\ \sqrt{Q_1}\,(\psi_1(\xi) - x_d) \\ \sqrt{R_1}\,(u_1 - u_d) \\ \vdots \\ \sqrt{Q_{N-1}}\,(\psi_{N-1}(\xi) - x_d) \\ \sqrt{R_{N-1}}\,(u_{N-1} - u_d) \\ \sqrt{Q_f}\,(\psi_N(\xi) - x_f) \end{bmatrix},$$

for some desired control u_d, desired state x_d, and desired final state x_f. Since $R_k > 0$ the Jacobian $\partial g(\xi)$ is guaranteed to be full rank and one can apply a GN iterative method directly to update $\xi \to \xi + \delta\xi$ where $\delta\xi = -(\partial_\xi g^T \partial_\xi g)^{-1} \partial_\xi g^T g$. In addition, the Jacobian has a lower-triangular structure that can be exploited in the Cholesky GN solution.

4. *Direct Collocation*: Implement a nonlinear programming strategy using direct transcription/collocation (as explained in the notes) to one of the two provided models (car or arm), or to a model of your choice. This can be accomplished by defining the cost and constraints and finding a solution using Matlab `fmincon`. See example `trajopt_sqp_car.m`. Obstacles should be added as inequality constraints.

   Note: upload your code to the File upload link and in addition attach a printout of your code and *plots* (annotated if necessary) to your homework solutions.