1 Continuous Systems with Terminal Constraints

Consider the cost
\[ J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt, \]
subject to \( q \) constraints
\[ \psi(x(t_f), t_f) = 0 \]
and the dynamics
\[ \dot{x}(t) = f(x(t), u(t), t), \quad t_0 \text{ and } x(t_0) \text{ are given}. \]

It will be useful to employ the shorthand notation \( f \equiv f(x(t), u(t), t) \), etc...
Sometimes, \( f \) (or any other function) could also be without arguments, i.e. \( f \equiv f(x(t), u(t), t) \).

To obtain the necessary conditions, form the augmented cost
\[ J_a = \phi(t_f) + \nu^T \psi(t_f) + \int_{t_0}^{t_f} \{ L(x, u, t) + \lambda^T(t)[f(x, u, t) - \dot{x}] \} dt. \]

Let \( \Phi = \phi + \nu^T \psi \) and define the Hamiltonian \( H \) by
\[ H(x, u, \lambda, t) = L(x, u, t) + \lambda^T(t)f(x, u, t). \]

Taking variations with respect to all variables including final time \( t_f \) we obtain
\[ \delta J_a = \left( \frac{\partial \Phi}{\partial t} + L + \lambda^T \dot{x} \right) \delta t_f + \int_{t_0}^{t_f} \left( \frac{\partial x}{\partial x} H + \frac{\partial u}{\partial u} L \right) \delta x + \frac{\partial u}{\partial u} H \cdot \delta u - \lambda^T \delta \dot{x} \right) dt. \]

Integrating by parts and using the relationship
\[ \delta x_f = \delta x(t_f) + \dot{x} \delta t_f, \]
we obtain
\[ \delta J_a = \left( \left[ \frac{\partial t}{\partial H} + L + \lambda^T \dot{x} \right] \delta t_f + \left[ \frac{\partial x}{\partial x} H - \lambda^T \right] \delta x_f \right)_{t=t_f} + \int_{t_0}^{t_f} \left( \frac{\partial x}{\partial x} H + \lambda^T \right) \delta x + \frac{\partial u}{\partial u} H \cdot \delta u \right) dt. \]

Since all variations are arbitrary and independent the necessary conditions become
\begin{align*}
\dot{\lambda}^T &= -\frac{\partial x}{\partial x} H = -\lambda^T \frac{\partial x}{\partial x} f - \partial x L, \quad (1) \\
\lambda(t_f)^T &= \left. \frac{\partial x}{\partial x} \Phi \right|_{t=t_f} = \left( \frac{\partial x}{\partial x} \phi + \nu^T \frac{\partial x}{\partial x} \psi \right)_{t=t_f}, \quad (2) \\
\frac{\partial u}{\partial u} H &= \lambda^T \frac{\partial u}{\partial u} f + \frac{\partial u}{\partial u} L = 0, \quad (3) \\
\left( \frac{\partial t}{\partial H} + L + \lambda^T \dot{x} \right)_{t=t_f} &= \left( \frac{\partial \Phi}{\partial t} + L \right)_{t=t_f} = 0, \quad (4)
\end{align*}
where
\[ \frac{d\Phi}{dt} = \partial_t \Phi + \partial_x \Phi \cdot \dot{x}. \]

After substituting the expression for \( \lambda(t_f) \) the necessary conditions are summarized according to:
\[ \dot{x} = f(x, u, t) \] \tag{5}
\[ \dot{\lambda} = -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L, \] \tag{6}
\[ \nabla_u H = \nabla_u f \cdot \lambda + \nabla_u L = 0, \] \tag{7}
\[ \lambda(t_f) = \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu, \] \tag{8}
\[ \left( \partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f \right)_{t=t_f} = 0, \] \tag{9}

When the final time \( t_f \) is fixed the last relationship (9) can be dropped.

**Hamiltonian conservation.** Note that whenever the Hamiltonian does not depend on time (that is when \( f \) and \( L \) do not depend on time)
\[ \partial_t H(x, u, \lambda, t) = 0 \]
then \( H \) is a conserved quantity along optimal trajectories \( x^*(t), u^*(t), \lambda^*(t) \), i.e. we have that
\[ H(x, u, \lambda, t) = \text{const} \]
for all \( t \in [t_0, t_f] \). Furthermore, in the special case when \( \partial_t \phi = 0 \) and \( \partial_t \psi = 0 \) the last condition [9] reduces to \( H(t) = 0 \).

**Minimum-time problems.** For minimum-time problems we have \( \phi = 0 \) and \( L = 1 \) so that condition [9] reduces to
\[ \left( \nu^T \partial_t \psi + \nabla_x \psi^T \cdot f \right)_{t=t_f} = 0, \]
which can be used along with the constraint \( \psi(x(t_f), t_f) = 0 \) to determine the multipliers \( \nu \) and final time \( t_f \).

**Solution Methods**

We are faced with solving the differential equations for \( t \in [t_0, t_f] \):
\[ \text{Euler-Lagrange (EL)} : \left( \begin{array}{c} \dot{x} \\ \dot{\lambda} \end{array} \right) = \left( \begin{array}{c} f(x, u, t) \\ -\nabla_x H \end{array} \right) \] \tag{12}
where \( u(t) \) is computed by minimizing \( H \) which corresponds to the condition
\[ \text{Control optimization} : \quad \nabla_u H = 0, \]

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which we assume can be solved and that \( u(t) \) is then expressed as a function of \( x(t) \) and \( \lambda(t) \), subject to the boundary constraints

\[
\begin{align*}
\psi(x(t_f), t_f) &= 0 \\
\lambda(t_f) &= \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu,
\end{align*}
\]

(13)

Transversality Conditions (TC):

\[
\left( \partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f \right)_{t=t_f} = 0,
\]

The following solution methods are applicable based on whether EL can be integrated in closed-form and whether TC can be solved in closed form:

- general: two-point boundary value problem (BVP) works with any EL and TC, the conditions are satisfied using a numerical “collocation” procedure
- EL integrable: pick \( \lambda(0) \) integrate from \( t_0 \) to \( t_f \) and solve TC as an implicit equality for the unknown \((\lambda(0), \nu)\). When final time \( t_f \) is free then solve for \((\lambda(0), \nu, t_f)\).
- EL integrable and TC solvable: closed-form solution.

**Example 1.** Minimum Control Effort Landing Consider a second order system with state \( x = (p, v) \in \mathbb{R}^4 \) where \( p \in \mathbb{R}^2 \) is the position and \( v \in \mathbb{R}^2 \) is the velocity. The system has a double integrator dynamics given by

\[
\begin{pmatrix}
\dot{p} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
v \\
u
\end{pmatrix},
\]

where \( u \in \mathbb{R}^2 \) is the acceleration control input. The system starts with known initial state \( x_0 = (p_0, v_0) \) and must “land” with a prescribed velocity \( v_f \) somewhere on a unit circle centered at the origin, i.e. the final configuration must satisfy \( \psi(x(t_f)) = 0 \), where

\[
\psi(x) = p^T p - 1.
\]

The objective function is the control effort given by

\[
L(x, u) = \frac{1}{2} \| u \|^2
\]

We start with the Hamiltonian, and the multipliers \( \lambda = (\lambda_p, \lambda_u) \)

\[
H = \frac{1}{2} u^T u + \lambda_p^T v + \lambda_u^T u,
\]

We have

\[
\begin{align*}
\dot{\lambda} &= -\nabla_x H \\
\nabla_u H &= 0
\end{align*}
\]

\[
\Rightarrow
\dot{\lambda}_p = 0, \quad \dot{\lambda}_u = -\lambda_p, \quad u = -\lambda_u,
\]

from which we get

\[
\ddot{u} = -\dot{\lambda}_u = \dot{\lambda}_p = 0,
\]

which means that the path \( p(t) \) is a cubic spline that can be written according to

\[
p(t_0 + t) = c_3 t^3 + c_2 t^2 + v_0 t + p_0,
\]

(14)
while the velocity is
\[ v(t_0 + t) = 3c_3 t^2 + 2c_2 t + v_0. \] (15)

Now from
\[ \lambda_p(t_f) = \nabla_p \psi(x(t_f)) \nu = 2p(t_f) \nu. \]

Note that above since the velocity is not present in the terminal constraint \( \psi \), then there is no additional condition on \( \lambda_v(t_f) \).

Now considering that \( \lambda_p(t_f) = \dot{u}(t_f) = 6c_3 \) the above is equivalent to
\[ 6c_3 = 2p(t_f) \nu. \]

Finally, assuming \( t_f \) is given we can solve for \( \nu, c_2, c_3 \) (5 unknowns) the implicit equations (5 equations):
\[
\begin{align*}
6c_3 - 2p(t_f) \nu = 0, \quad & \text{(16)} \\
p(t_f)^T p(t_f) - 1 = 0, \quad & \text{(17)} \\
v(t_f) - v_f = 0, \quad & \text{(18)}
\end{align*}
\]

where \( p(t_f) \) and \( v(t_f) \) are given by (14) and (15). Note that it is necessary that \( \nu \neq 0 \) to ensure that the constraint is satisfied.

Examples of the resulting trajectories from randomly initialized states are given. In all examples we have \( v_f = (0, 0) \)

![Diagram showing trajectories](image)

**Example 2.** Example: Zermelo’s problem (Bryson §2.7) Consider a ship with dynamics
\[
\begin{align*}
\dot{x} &= V \cos \theta + u(x, y) \quad \text{(19)} \\
\dot{y} &= V \sin \theta + v(x, y), \quad \text{(20)}
\end{align*}
\]

where \( (x, y) \) is the position, \( V \) is a constant velocity, \( \theta \) is the heading angle input and \( u \) and \( v \) denote velocity due to currents. The goal is to travel between points \( A \) and \( B \) in minimum time.
The Hamiltonian is

\[ H = \lambda_x (V \cos \theta + u) + \lambda_y (V \sin \theta + v) + 1. \]

The Euler-Lagrange equations are

\[ \dot{\lambda}_x = -\partial_x H = -\lambda_x \partial_x u - \lambda_y \partial_x v \]
\[ \dot{\lambda}_y = -\partial_y H = -\lambda_x \partial_y u - \lambda_y \partial_y v \]
\[ 0 = \partial_\theta H = V (-\lambda_x \sin \theta + \lambda_y \cos \theta) \Rightarrow \tan \theta = \frac{\lambda_y}{\lambda_x} \]

Since this is a minimum-time problem we have \( H = 0 \) and from (23) that

\[ \lambda_x = \frac{-\cos \theta}{V + u \cos \theta + v \sin \theta}, \quad \lambda_y = \frac{-\sin \theta}{V + u \cos \theta + v \sin \theta} \]

This leads to

\[ \dot{\theta} = \sin^2 \theta \partial_x v + \sin \theta \cos \theta (\partial_x u - \partial_y v) - \cos^2 \theta \partial_y u \]

Now, in order to reach \( B \) one has to select the start angle \( \theta_0 \) and the final time \( t_f \).

**Special Case.** For the special case when

\[ u = -V(y/h), \quad v = 0 \]

consider starting at \((x_0, y_0)\) with the goal to reach the origin \((0, 0)\). We have

\[ \dot{\lambda}_x = 0 \Rightarrow \lambda_x = \text{const} \]

and therefore

\[ \lambda_x = \frac{-\cos \theta}{V - V(y/h) \cos \theta} = \frac{-\cos \theta_f}{V} = \text{const} \Rightarrow \cos \theta = \frac{\cos \theta_f}{1 + (y/h) \cos \theta_f} \]

In the above, it turned out that it is convenient to work in terms of \( \theta_f \) rather than \( t_f \). The solution can be obtained analytically as

\[ x = \frac{h}{2} \left[ \sec \theta_f (\tan \theta_f - \tan \theta) - \tan \theta (\sec \theta_f - \sec \theta) + \log \frac{\tan \theta_f + \sec \theta_f}{\tan \theta + \sec \theta} \right], \]
\[ y = h(\sec \theta - \sec \theta_f), \]

from which one can compute the initial angle \( \theta \) and final angles \( \theta_f \) to achieve given final position \((x, y)\).

The computed path with initial conditions given by \( x_0 = 3.66 \) and \( y_0 = -1.86 \) with \( h = 1 \), \( V = .3 \) are given below
Example 3. Minimum Control Effort Landing with Optimal Time Consider the minimum control effort landing \([1]\) with free final time \(t_f\) and a cost function given by

\[
L(x, u) = b + \frac{1}{2} \| u \|^2,
\]

for some constant \(b > 0\) which controls the balance between penalizing total time and total control effort.

We need to add the third transversality condition from \((13)\)

\[
\partial_t \phi(t_f) + H(t_f) = 0,
\]

which in our case is

\[
b - \frac{1}{2} \| u(t_f) \|^2 + \dot{u}(t_f)^T v(t_f) = 0
\]

This can be solved along with the other five conditions to obtain the unknowns \(c_2, c_3, \nu, t_f\). Plots of computed trajectories with varying \(b\) are given below.