1 Variations of functions

We are interested in solving optimal control problems such as

$$\min J(x(\cdot), u(\cdot), t_f) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt$$

subject to $\dot{x} = f(x, u, t)$ and other constraints. The cost $J$ is called a functional, i.e. it is a function of functions since the trajectories $x(\cdot)$ and $u(\cdot)$ are functions of time. It is possible to optimize a functional in a similar way we optimize a regular function. In particular, there is a functional analog to the necessary conditions for a minimum of a function $g$ given by $\nabla g = 0$. This analog is

$$\nabla g = 0 \iff \text{variation of a functional } \delta J = 0$$

Next, define the change in a functional, after varying $x(t)$ by $\delta x(t)$ at each $t$, by

$$\Delta J(x(\cdot), \delta x(\cdot)) = J(x(\cdot) + \delta x(\cdot)) - J(x(\cdot))$$

The variation $\delta J(x(\cdot), \delta x(\cdot))$ is a linear function of $\delta x(\cdot)$ defined by the following relationship

$$\Delta J(x(\cdot), \delta x(\cdot)) = \delta J(x(\cdot), \delta x(\cdot)) + o(\|\delta x\|),$$

where the small-o notation was used. For a positive integer $p$ and a function $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ we have

$$h(x) = o(\|x\|^p),$$

if $\lim_{k \to \infty} \frac{h(x_k)}{\|x_k\|^p} = 0$ for all sequences $x_k$ such that $x_k \to 0$ and $x_k \neq 0$ for all $k$.

In other words, roughly speaking, if $h(x)$ denotes the the second and higher order terms in $\Delta J$, then $h(x)$ has second and higher-order multiples of $\delta x$ and thus $h(x)/\|\delta x\|$ is at least proportional to $\delta x$ which tends to zero as $\delta x \to 0$. 

\[x(t)\]

\[x(\cdot)\]

\[x + \delta x\]

\[x(\cdot)\]

\[x - \delta x\]

\[\delta x\]

\[t_0\]

\[t_f\]
Similarly to standard function optimization, we can use the argument that at an optimum \( x^*(\cdot) \) we have
\[
\Delta J(x^*(\cdot), \delta x(\cdot)) \geq 0
\]
since any change of the cost away from optimum must be positive. Taking \( \delta x \to 0 \) this implies that
\[
\delta J(x^*(\cdot), \delta x(\cdot)) \geq 0.
\]
But this must also hold for variations in the direction \(-\delta x(\cdot)\), i.e.
\[
\delta J(x^*(\cdot), -\delta x(\cdot)) \geq 0 \quad \Rightarrow \quad \delta J(x^*(\cdot), \delta x(\cdot)) \leq 0,
\]
by the linearity of \( \delta J \). Both conditions can only be true when
\[
\delta J(x^*(\cdot), \delta x(\cdot)) = 0,
\]
for any \( \delta x(\cdot) \).

2 The Euler-Lagrange Equations

Consider the cost function
\[
J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t))dt.
\]
Its variation becomes
\[
\delta J = \int_{t_0}^{t_f} \left[ g_{x}(x, \dot{x}) \delta x + g_{\dot{x}}(x, \dot{x}) \delta \dot{x} \right] dt,
\]
since the variation corresponds to taking first-order terms (i.e. derivatives) of the functions at time \( t \). Integrate by parts (recall \( \int u \dot{v} = uv - \int \dot{u}v \)) to get
\[
\delta J = \int_{t_0}^{t_f} \left[ g_{x}(x, \dot{x}) \delta x - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \delta x \right] dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f)) \delta x(t_f)
\]
**Fixed boundary conditions.** If \( x(t_0) \) and \( x(t_f) \) are given then \( \delta x(0) = \delta x(t_f) = 0 \) and
\[
\delta J = \int_{t_0}^{t_f} \left[ g_{x}(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \right] \delta x(t)dt
\]
Since \( \delta x(t) \) are arbitrary and independent then \( \delta J = 0 \) only when
\[
g_{x}(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) = 0,
\]
which are called the Euler-Lagrange equations (EL).

**Free boundary conditions.** When \( x(t_f) \) is not fixed, in addition to the EL equation, the following must hold
\[
g_{\dot{x}}(x(t_f), \dot{x}(t_f)) = 0.
\]
2.1 Example: shortest path curve

Consider a one-dimensional problem with state $x(t)$ and the cost function

$$J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2} dt,$$

which in fact corresponds to the length of a curve in the $(x,t)$ plane. The goal is to compute the shortest length curve between two given points $(x_0, t_0)$ and $(x_f, t_f)$.

Applying the EL equations we get

$$\frac{d}{dt} \frac{\dot{x}}{(1 + \dot{x}^2)^{1/2}} = 0 \iff \frac{\ddot{x}}{(1 + \dot{x}^2)^{3/2}} = 0,$$

which is satisfied when $\ddot{x} = 0$ or when

$$x(t) = c_1 t + c_0,$$

i.e. when $x(t)$ is a straight line. Let $t_0 = 0$ and $t_f = 1$. It is easy to see that $c_0 = x_0$ and $c_1 = x_f - x_0$.

2.2 Particle in 3-D

Let $g(x, \dot{x}, t) = \frac{1}{2} m \| \dot{x} \|^2 - V(x)$, where $m$ denotes the mass and $V$ denotes the potential energy of a particle with position $x \in \mathbb{R}^3$. The function $g$ is actually the Lagrangian of the particle and the EL equations lead to

$$m \ddot{x} = -\nabla_x V,$$

which is simply Newton’s law. This is one of the simplest examples that illustrates that Lagrangian mechanics can be considered as a special case of optimal control.

3 Free final-time

When the final time $t_f$ is allowed to vary, the variation of $J$ is expressed as

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x^*(t), \dot{x}^*(t), t) \delta x + \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \delta \dot{x} \right] dt + g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f + g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f,$$

where we employ the spatially-optimized trajectory $x^*(t)$ instead of just $x(t)$ to signify that after finding the optimal path without considering $\delta t_f$, then variations of $t_f$ infinitesimally contribute to the cost by the term

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f$$

which is simply the cost at the last points multiplied by the time variations: think of it as a first-order approximation to the integral $\int_{t_f}^{t_f + \delta t_f} g(x^*, \dot{x}^*, t) dt$ which is what must be added to the cost when varying time.
Define the total space-time variation \( \delta x_f \) by
\[
\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f,
\]
i.e. this variation combines the effects of varying the final state by keeping time \( t_f \) fixed, and then by varying \( t_f \) (See figure above).

We have
\[
\delta J = \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t)\delta x - \frac{d}{dt}g_x(x^*, \dot{x}^*, t)\delta x \right] dt + \left( g_x(x^*, \dot{x}^*, t_f)\delta x + \left[ g(x^*, \dot{x}^*, t_f) - g_x(x^*, \dot{x}^*, t_f)\dot{x} \right] \delta t_f \right)_{t=t_f}.
\]

Unrelated \( t_f \) and \( x(t_f) \). We next consider the case when \( t_f \) and \( x(t_f) \) are unrelated in which we have
\[
g_x(t_f) = 0, \quad \left[ g(t_f) - g_x(t_f)\dot{x} \right] \delta t_f = 0 \quad \Rightarrow \quad g(t_f) = 0
\]

Function \( \Theta(t_f) \). We next consider the case when the boundary constraint is given by
\[
x(t_f) = \Theta(t_f)
\]

Variations \( \delta t_f \) and \( \delta x_f \) are related by
\[
\delta x_f = \frac{d\Theta}{dt}\delta t_f
\]
and hence the necessary conditions become
\[
g_x(t_f) \left( \frac{d\Theta}{dt} - \dot{x}^* \right)_{t=t_f} + g(t_f) = 0,
\]
which are called trasversality conditions.

4 Differential Constraints

Consider the optimization of
\[
J = \int_{t_0}^{t_f} g(x, \dot{x}, t)dt
\]
where $x \in \mathbb{R}^{n+m}$ subject to

$$f(x(t), \dot{x}(t), t) = 0,$$

where $f = (f_1, \ldots, f_n)$ are $n$ constraints, and $x(t_f)$ and $t_f$ are fixed. To obtain the necessary conditions, we define the Lagrangian multipliers $\lambda : [t_0, t_f] \rightarrow \mathbb{R}^n$ and the augmented cost

$$J_a = \int_{t_0}^{t_f} \left\{ g(x, \dot{x}, t) + \lambda^T f(x, \dot{x}, t) \right\} dt$$

Taking variations

$$J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x] \delta x(t) + [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta \dot{x} + \delta \lambda^T f \right\} dt$$

Integrating by parts we get

$$J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x] \delta x(t) + [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta \dot{x} + \delta \lambda^T f \right\} dt.$$

If we define the augmented cost $g_a$ by

$$g_a = g + \lambda^T f$$

the Euler-Lagrange equations become

$$\frac{\partial g_a}{\partial x} - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{x}} = 0$$

which along with the constraint

$$f(x, \dot{x}, t) = 0$$

constitute the necessary conditions.

5 General Boundary Constraints

Let $x \in \mathbb{R}^n$ and consider the optimization of

$$J(x(\cdot), t_f) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$

subject to the free final time $t_f$ and general boundary conditions

$$\psi(x(t_f), t_f) = 0,$$

where $\psi$ is a vector of $m$ functions. To obtain the necessary conditions define the augmented cost

$$J_a = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt,$$

where $\nu \in \mathbb{R}^m$. Let

$$w(x(t_f), \nu, t_f) = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f)$$
Taking variations 

\[ \delta J_a = w_x(t_f) \delta x_f + w_{t_f}(t_f) \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x \cdot \delta x(t) + g_{x\dot{x}} \cdot \delta \dot{x}(t)] dt + g(t_f) \delta t_f, \]

where the last term is due to cost accrued from final time variations. Using integration by parts as well as the total variation definition \( \delta x_f = \delta x(t_f) + \dot{x}(t_f) \delta t_f \) we have

\[ \int_{t_0}^{t_f} g_x \cdot \delta \dot{x}(t) dt = g_x(t_f) \delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt} g_x \cdot \delta x(t) \]

\[ = g_x(t_f)(\delta x_f - \dot{x}(t_f)) - \int_{t_0}^{t_f} \frac{d}{dt} g_x \cdot \delta x(t) \]

which results in

\[ \delta J_a = [w_x(t_f) + g_x] \delta x_f + [w_{t_f}(t_f) + g(t_f) - g_{x\dot{x}}(t_f) \dot{x}(t_f)] \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x - \frac{d}{dt} g_{x\dot{x}}] \delta x(t) dt, \]

The necessary conditions require that \( \delta J_a = 0 \) for arbitrary \( \delta x(t), \delta \nu \) which is only possible if the following necessary conditions hold:

\[ \nabla_x w(x(t_f), \nu, t_f) + \nabla_{x\dot{x}} g(x(t_f), \dot{x}(t_f), t_f) = 0, \quad (1) \]

\[ \frac{\partial}{\partial t_f} w(x(t_f), \nu, t_f) + g(x(t_f), \dot{x}(t_f), t_f) - \nabla_{x\dot{x}} g(x(t_f), \dot{x}(t_f), t_f)^T \dot{x}(t_f) = 0, \quad (2) \]

\[ \psi(x(t_f), t_f) = 0 \quad (3) \]

\[ \nabla_x g(x(t), \dot{x}(t), t) - \frac{d}{dt} \nabla_{x\dot{x}} g(x(t), \dot{x}(t), t) = 0, \quad t \in (t_0, t_f). \quad (4) \]