1 Variations of functions

We are interested in solving optimal control problems such as

$$\min J(x(\cdot), u(\cdot), t_f) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt$$

subject to \( \dot{x} = f(x, u, t) \) and other constraints. The cost \( J \) is called a functional, i.e. it is a function of functions since the trajectories \( x(\cdot) \) and \( u(\cdot) \) are functions of time. It is possible to optimize a functional in a similar way we optimize a regular function. In particular, there is a functional analog to the necessary conditions for a minimum of a function \( g \) given by

$$\nabla g = 0$$

$$\iff \delta J = 0$$

Next, define the change in a functional, after varying \( x(t) \) by \( \delta x(t) \) at each \( t \), by

$$\Delta J(x(\cdot), \delta x(\cdot)) = J(x(\cdot) + \delta x(\cdot)) - J(x(\cdot))$$

The variation \( \delta J(x(\cdot), \delta x(\cdot)) \) is a linear function of \( \delta x(\cdot) \) defined by the following relationship

$$\Delta J(x(\cdot), \delta x(\cdot)) = \delta J(x(\cdot), \delta x(\cdot)) + o(\|\delta x\|),$$

where the small-o notation was used. For a positive integer \( p \) and a function \( h : \mathbb{R}^n \to \mathbb{R}^m \) we have

$$h(x) = o(\|x\|^p),$$

if \( \lim_{k \to \infty} \frac{h(x_k)}{\|x_k\|^p} = 0 \) for all sequences \( x_k \) such that \( x_k \to 0 \) and \( x_k \neq 0 \) for all \( k \).

In other words, roughly speaking, if \( h(x) \) denotes the the second and higher order terms in \( \Delta J \), then \( h(x) \) has second and higher-order multiples of \( \delta x \) and thus \( h(x)/\|\delta x\| \) is at least proportional to \( \delta x \) which tends to zero as \( \delta x \to 0 \).
Similarly to standard function optimization, we can use the argument that at an optimum $x^*(\cdot)$ we have
\[ \Delta J(x^*(\cdot), \delta x(\cdot)) \geq 0 \]
since any change of the cost away from optimum must be positive. Taking $\delta x \to 0$ this implies that
\[ \delta J(x^*(\cdot), \delta x(\cdot)) \geq 0. \]
But this must also hold for variations in the direction $-\delta x(\cdot)$, i.e.
\[ \delta J(x^*(\cdot), -\delta x(\cdot)) \geq 0 \implies \delta J(x^*(\cdot), \delta x(\cdot)) \leq 0, \]
by the linearity of $\delta J$. Both conditions can only be true when
\[ \delta J(x^*(\cdot), \delta x(\cdot)) = 0, \]
for any $\delta x(\cdot)$.

2 The Euler-Lagrange Equations

Consider the cost function
\[ J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t))dt. \]
Its variation becomes
\[ \delta J = \int_{t_0}^{t_f} [g_x(x, \dot{x})\delta x + g_{\dot{x}}(x, \dot{x})\delta \dot{x}] dt, \]
since the variation corresponds to taking first-order terms (i.e. derivatives) of the functions at time $t$. Integrate by parts (recall $\int u\dot{v} = uv - \int \dot{u}v$) to get
\[ \delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x})\delta x - \frac{d}{dt}g_{\dot{x}}(x, \dot{x})\delta \dot{x} \right] dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f))\delta x(t_f) \]

**Fixed boundary conditions.** If $x(t_0)$ and $x(t_f)$ are given then $\delta x(0) = \delta x(t_f) = 0$ and
\[ \delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) - \frac{d}{dt}g_{\dot{x}}(x, \dot{x}) \right] \delta x(t)dt \]
Since $\delta x(t)$ are arbitrary and independent then $\delta J = 0$ only when
\[ g_x(x, \dot{x}) - \frac{d}{dt}g_{\dot{x}}(x, \dot{x}) = 0, \]
which are called the Euler-Lagrange equations (EL).

**Free boundary conditions.** When $x(t_f)$ is not fixed, in addition to the EL equation, the following must hold
\[ g_{\dot{x}}(x(t_f), \dot{x}(t_f)) = 0. \]
2.1 Example: shortest path curve

Consider a one-dimensional problem with state $x(t)$ and the cost function

$$J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2} \, dt,$$

which in fact corresponds to the length of a curve in the $(x, t)$ plane. The goal is to compute the shortest length curve between two given points $(x_0, t_0)$ and $(x_f, t_f)$.

Applying the EL equations we get

$$\frac{d}{dt} \frac{\dot{x}}{(1 + \dot{x}^2)^{1/2}} = 0 \quad \Leftrightarrow \quad \frac{\ddot{x}}{(1 + \dot{x}^2)^{3/2}} = 0,$$

which is satisfied when $\ddot{x} = 0$ or when

$$x(t) = c_1 t + c_0,$$

i.e. when $x(t)$ is a straight line. Let $t_0 = 0$ and $t_f = 1$. It is easy to see that $c_0 = x_0$ and $c_1 = x_f - x_0$.

2.2 Particle in 3-D

Let $g(x, \dot{x}, t) = \frac{1}{2} m \|\dot{x}\|^2 - V(x)$, where $m$ denotes the mass and $V$ denotes the potential energy of a particle with position $x \in \mathbb{R}^3$. The function $g$ is actually the Lagrangian of the particle and the EL equations lead to

$$m \ddot{x} = -\nabla_x V,$$

which is simply Newton’s law. This is one of the simplest examples that illustrates that Lagrangian mechanics can be considered as a special case of optimal control.

3 Free final-time

When the final time $t_f$ is allowed to vary, the variation of $J$ is expressed as

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t) \delta x - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \delta \dot{x} \right] dt + g_{\ddot{x}}(x^*(t_f), \dot{x}^*(t_f), t) \delta \dot{x}(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f,$$

where we employ the spatially-optimized trajectory $x^*(t)$ instead of just $x(t)$ to signify that after finding the optimal path without considering $\delta t_f$, then variations of $t_f$ infinitesimally contribute to the cost by the term

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f$$

which is simply the cost at the last points multiplied by the time variations: think of it as a first-order approximation to the integral $\int_{t_f}^{t_f + \delta t_f} g(x^*, \dot{x}^*, t) dt$ which is what must be added to the cost when varying time.
Define the total space-time variation $\delta x_f$ by

$$\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f,$$

i.e. this variation combines the effects of varying the final state by keeping time $t_f$ fixed, and then by varying $t_f$ (See figure above).

We have

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t) \delta x - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \delta x \right] dt + \left( g_{\dot{x}}(x^*, \dot{x}^*, t_f) \delta x_f + [g(x^*, \dot{x}^*, t_f) - g_x(x^*, \dot{x}^*, t_f) \dot{x}] \delta t_f \right)_{t=t_f}.$$

**Unrelated $t_f$ and $x(t_f)$.** We next consider the case when $t_f$ and $x(t_f)$ are unrelated in which we have

$$g_{\dot{x}}(t_f) = 0, \quad [g(t_f) - g_x(t_f)\dot{x}] \delta t_f = 0 \quad \Rightarrow \quad g(t_f) = 0$$

**Function $\Theta(t_f)$.** We next consider the case when the boundary constraint is given by

$$x(t_f) = \Theta(t_f)$$

Variations $\delta t_f$ and $\delta x_f$ are related by

$$\delta x_f = \frac{d\Theta}{dt} \delta t_f$$

and hence the necessary conditions become

$$g_{\dot{x}}(t_f) \left[ \frac{d\Theta}{dt} - \dot{x}^* \right]_{t=t_f} + g(t_f) = 0,$$

which are called **trasversality conditions.**

### 4 Differential Constraints

Consider the optimization of

$$J = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$
where \( x \in \mathbb{R}^{n+m} \) subject to

\[
    f(x(t), \dot{x}(t), t) = 0,
\]

where \( f = (f_1, \ldots, f_n) \) are \( n \) constraints, and \( x(t_f) \) and \( t_f \) are fixed. To obtain the necessary conditions, we define the Lagrangian multipliers \( \lambda : [t_0, t_f] \rightarrow \mathbb{R}^n \) and the augmented cost

\[
    J_a = \int_{t_0}^{t_f} \left\{ g(x, \dot{x}, t) + \lambda^T f(x, \dot{x}, t) \right\} dt
\]

Taking variations

\[
    J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x] \delta x(t) + [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta \dot{x} + \delta \lambda^T f \right\} dt
\]

Integrating by parts we get

\[
    J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x - \frac{d}{dt} [g_{\dot{x}} + \lambda^T f_{\dot{x}}]] \delta x + \delta \lambda^T f \right\} dt.
\]

If we define the augmented cost \( g_a \) by

\[
    g_a = g + \lambda^T f
\]

the Euler-Lagrange equations become

\[
    \frac{\partial g_a}{\partial x} - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{x}} = 0
\]

which along with the constraint

\[
    f(x, \dot{x}, t) = 0
\]

constitute the necessary conditions.

5 General Boundary Constraints

Let \( x \in \mathbb{R}^n \) and consider the optimization of

\[
    J(x(\cdot), t_f) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt
\]

subject to the free final time \( t_f \) and general boundary conditions

\[
    \psi(x(t_f), t_f) = 0,
\]

where \( \psi \) is a vector of \( m \) functions. To obtain the necessary conditions define the augmented cost

\[
    J_a = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt,
\]

where \( \nu \in \mathbb{R}^m \). Let

\[
    w(x(t_f), \nu, t_f) = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f)
\]

5
Taking variations

\[ \delta J_a = w_x(t_f) \delta x_f + w_{t_f}(t_f) \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x \cdot \delta x(t) + g_\dot{x} \cdot \dot{\delta x}(t)] dt + g(t_f) \delta t_f, \]

where the last term is due to cost accrued from final time variations. Using integration by parts as well as the total variation definition \( \delta x_f = \delta x(t_f) + \dot{x}(t_f) \delta t_f \) we have

\[ \int_{t_0}^{t_f} g_\dot{x} \cdot \dot{\delta x}(t) dt = g_\dot{x}(t_f) \delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt} g_\dot{x} \cdot \delta x(t) \]

which results in

\[ \delta J_a = [w_x(t_f) + g_\dot{x}] \delta x_f + [w_{t_f}(t_f) + g(t_f) - g_\dot{x}(t_f) \dot{x}(t_f)] \delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} \left[ g_x - \frac{d}{dt} g_\dot{x} \right] \delta x(t) dt, \]

The necessary conditions require that \( \delta J_a = 0 \) for arbitrary \( \delta x(t), \delta \nu \) which is only possible if the following necessary conditions hold:

\[ \nabla_x w(x(t_f), \nu, t_f) + \nabla_\dot{x} g(x(t_f), \dot{x}(t_f), t_f) = 0, \tag{1} \]

\[ \frac{\partial}{\partial t_f} w(x(t_f), \nu, t_f) + g(x(t_f), \dot{x}(t_f), t_f) - \nabla_\dot{x} g(x(t_f), \dot{x}(t_f), t_f)^T \dot{x}(t_f) = 0, \tag{2} \]

\[ \psi(x(t_f), t_f) = 0 \tag{3} \]

\[ \nabla_x g(x(t), \dot{x}(t), t) - \frac{d}{dt} \nabla_\dot{x} g(x(t), \dot{x}(t), t) = 0, \quad t \in (t_0, t_f). \tag{4} \]